THE BURNING NUMBER

FINAL REPORT

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Let's say you wake up one day and you decide that you want to burn a forest. However, you do not have unlimited resources and you want to burn it as fast as possible. You would then like to know what is the fastest way possible to burn this forest. Fortunately, you have heard that mathematicians have recently been studying the so-called burning number of graphs.

A burning process of a graph G goes as follows :
At time $t = 0$, you choose a vertex of G to burn.
At time $t = 1$, every vertex spreads its fire to all of its neighbours in G, then you burn another vertex.
$\cdots$
At time $t = k - 1$, every vertex spreads its fire to all of its neighbours, then you burn another vertex.
At time $t = k$, every vertex is already burnt.

The burning number of G (denoted $B(G)$) is the minimal k possible in any burning process of G. Clearly the burning number exists, since we could just choose to burn each vertex one per one until everything is burnt.

Let's now be more formal. Given a connected graph $G = (V, E)$, we can see it as a metric space with distance between $u, v \in V$ as $d(u, v)$ equals the length of the shortest path between u and v in G. You can convince yourself that d is indeed a metric.

Denote $B_r(u) = \{v \in V | d(u, v) \leq r\}$.
$B(G)$ is then the minimal k such that there exist $u_0, \cdots, u_{k-1} \in V$ with
$V = B_0(u_0) \cup \cdots \cup B_{k-1}(u_{k-1})$.

The most important conjecture about the burning number is that for any connected graph G, $B(G) \leq \lceil \sqrt{|G|} \rceil$. We know this is true when G is a path and we believe that paths are the hardest graphs to burn, which leads us to this conjecture.

In this paper, we will study a similar but different topic. Given three non-negative numbers r_1, r_2, r_3, for which classes of graphs is it always possible to cover them with balls of radii r_1, r_2, r_3?

By covering a graph $G = (V, E)$ with balls of radii r_1, r_2, r_3, we mean that there exist vertices $u_1, u_2, u_3 \in V$ such that $V = B_{r_1}(u_1) \cup B_{r_2}(u_2) \cup B_{r_3}(u_3)$.

We will now start going through some notation that will help us proving the **Theorem 1** of this report.

Notation 1. *If G is a graph, we will denote its vertex set by $V(G)$ and its edge set by $E(G)$, so that $G = (V(G), E(G))$.*

Notation 2. *If G is a graph and $k \in \mathbb{R}$, writing $|G| = k$ means that $|V(G)| = k$.*

Notation 3. *If G is a graph, writing $u \in G$ means that $u \in V(G)$.*

Notation 4. *For a connected graph G and $u, v \in G$, we denote the shortest subpath of G with endpoints u, v by $P_{u,v}$. We can denote $P_{u,v}^G$ if we want to specify that we are refering to the shortest subpath of G.*

When we are dealing with graphs and subgraphs, it can be convenient to introduce this notation when it is ambiguous which graph we are working with:

Notation 5. *If G is a graph, we denote $B_r^G(a) = \{u \in G \mid d_G(a, u) \leq r\}$.*

In this report, we will often work with trees that have 5 leaves or less. These trees can all be represented as in the next figure, where v_1, v_2, v_4, v_6, v_7 are the leaves of each branch, $A = \{u_{123}, u_{345}, u_{567}\}$ are the vertices of degree greater or equal to 3, and $l_1, \cdots, l_7$ denote the vertex sets of each branches excluding $u_{123}, u_{345}, u_{567}$, so $v_i \in l_i$ for $i \in \{1, 2, 4, 6, 7\}$. Notice that it would be possible that some of $l_1, \cdots, l_7$ are empty. In such a case, we will simply establish that v_1 is equal to its corresponding closest vertex in A in the Figure 1.

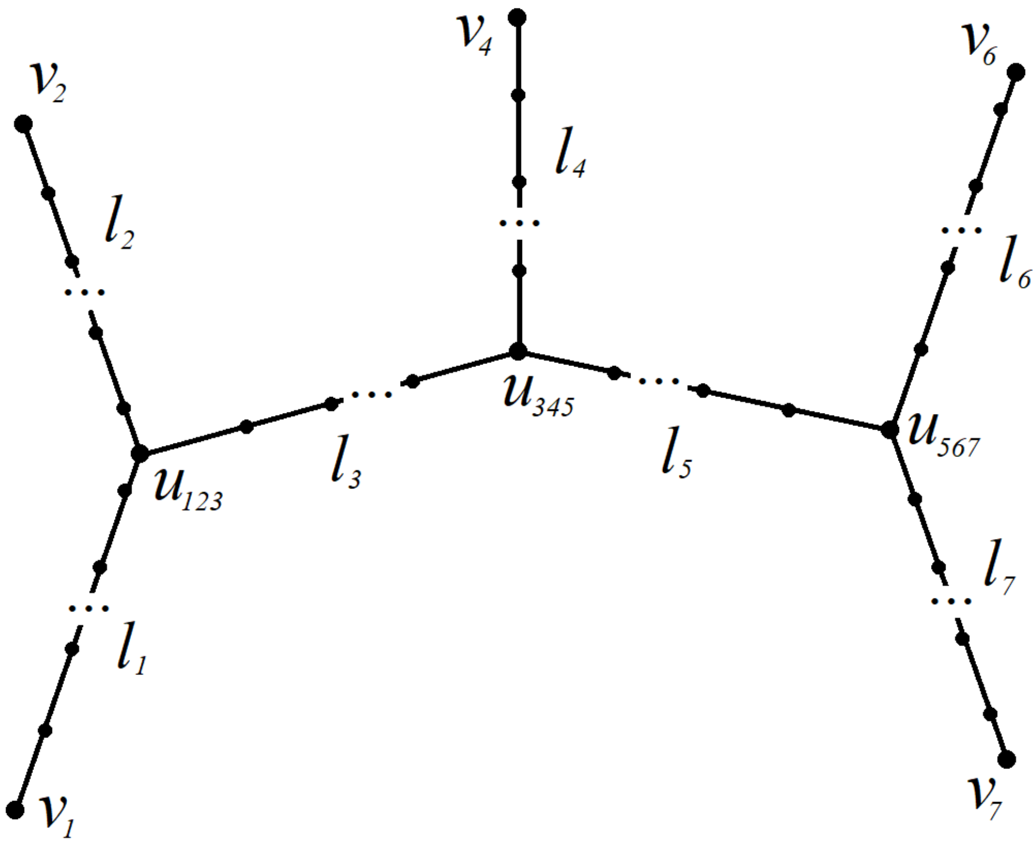

Figure 1 : Our model of a 5-leaves tree

For convenience, we will introduce another notation:

Notation 6. *In the figure above, we denote* $l_{i_1,\cdots,i_k} = l_{i_1} \cup \cdots \cup l_{i_k} \cup \{u \in A \mid u$ *is neighbour to at least two vertices in* $l_{i_1} \cup \cdots \cup l_{i_k}\}$.

For example, $l_{1,2,3} = l_1 \cup \{u_{123}\} \cup l_2 \cup l_3$ and $l_{4,5,6,7} = l_4 \cup \{u_{345}\} \cup l_5 \cup \{u_{567}\} \cup l_6 \cup l_7$.

Notice here that l_i remains the same set that it previously was for each $i \in \{1, \cdots, 7\}$, so that doesn't create any problem.

We will look at connected graphs as metric spaces with distances between vertices being the length of the shortest path linking them and we will study under what circumstances it is possible to cover such graphs with certain balls of certain radii.

We will now introduce some definitions.

Definition 1. *We say that a set* $\{r_1, \ldots, r_k\}$ *covers a graph* G *if there exist* $u_1, \ldots, u_k \in G$ *such that* $V(G) = B_{r_1}(u_1) \cup \ldots \cup B_{r_k}(u_k)$*.*

Definition 2. *We call a set* $\{r_1, \ldots, r_k\}$ *a cover if it covers every connected graph* G *with* $|G| = 2(r_1 + \ldots + r_k) + k$*.*

Definition 3. *We say that a set* $\{r_1, \cdots, r_k\}$ *covers a vertex subset* V *of a graph* G *if there exist* $u_1, \cdots, u_k \in G$ *such that* $V \subseteq B_{r_1}(u_1) \cup \cdots \cup B_{r_k}(u_k)$*.*

Now, we are ready to introduce the main theorem of this paper that we are trying to prove:

Theorem 1. *If* $0 \leq r_1 \leq r_2 \leq r_3$ *and* $2r_2 \leq r_3$*, then* $\{r_1, r_2, r_3\}$ *is a cover.*

In order to prove this, we will need a couple of results first.

Proposition 1. *Let* $0 \leq r_1 \leq \cdots \leq r_k$*,* T *a tree, and* $T' \subseteq T$ *be connected. If* $\{r_1, \cdots, r_k\}$ *is a cover for* T*, then it is a cover for* T'*.*

Proof. Pick $u_1, \cdots, u_k \in T$ such that $V(T) = B_{r_1}(u_1) \cup \cdots \cup B_{r_k}(u_k)$. Let $1 \leq i \leq k$.
If $V(T') \cap B_{r_i}(u_i)$ is nonempty, then set $v_i \in V(T') \cap B_{r_i}(u_i)$ such that $d_T(v_i, u_i)$ is minimal. v_i is uniquely defined and either $u_i = v_i$ or v_i is a leaf of T'.
Let $I = \{1 \leq i \leq k \mid V(T') \cap B_{r_i}(u_i)$ is nonempty$\}$.
Let $u \in T' \subseteq T$. Let $1 \leq i \leq k$ such that $u \in B_{r_i}(u_i)$. We know that $i \in I$.
If $u_i \in T'$, then $u \in B_{r_i}(v_i) = B_{r_i}(u_i)$.
If $u_i \notin T'$, then since T is a tree, there is a unique path $P \subseteq T$ with endpoints u_i and u and $v_i \in P$. So $d_{T'}(u, v_i) \leq d_T(u, u_i)$, thus $u \in B_{r_i}(v_i)$.
We see that $\forall u \in T'$, $\exists i \in I$ such that $B_{r_i}(v_i)$, so $V(T') = \cup_{i \in I} B_{r_i}(v_i)$.

□

Proposition 2. *Let* $0 \leq r_1 \leq \cdots \leq r_k$ *and* G *a graph. If* $H \subseteq \{r_1, \cdots, r_k\}$ *is a cover for* G*, then* $\{r_1, \cdots, r_k\}$ *is a cover for* G*.*

Proof. Assume that $H \subseteq \{r_1, \cdots, r_k\}$ is a cover for G, then let $H' = \{i_1, \cdots, i_{|H|}\} \subseteq \{1, \cdots, k\}$ and $\{u_1, \cdots, u_{|H|}\} \subseteq V(G)$ such that $V(G) = B_{r_{i_1}}(u_1) \cup \cdots \cup B_{r_{i_{|H|}}}(u_{|H|})$. Then $V(G) = (\cup_{j \in H'} B_{i_j}(u_j)) \cup (\cup_{j \in \{1, \cdots, k\} \setminus H'} B_{i_j}(u_1))$, so $\{r_1, \cdots, r_k\}$ covers G.

□

Proposition 3. *Let $0 \leq r_1 \leq \cdots \leq r_k$ and G, H two graphs such that $V(G) = V(H)$ and $E(H) \subseteq E(G)$. If $\{r_1, \cdots, r_k\}$ is a cover for H, then it is a cover for G.*

Proof. Let $\{r_1, \cdots, r_k\}$ be a cover for H, then let $u_1, \cdots, u_k \in G$ such that $V(H) = B^H_{r_1}(u_1) \cup \cdots \cup B^H_{r_k}(u_k)$.
Let $i \in \{1, \cdots, k\}$, then let $u \in B^H_{r_i}(u_i)$. $P^H_{u,u_i} \subseteq E(G)$, so the shortest u, u_i-path in G has size at most $|P_{u,u_i}|$, so $d_G(u, u_i) \leq d_H(u, u_i) \leq r_i$, so $u \in B^G_{r_i}(u_i)$. Since u was chosen arbitrarily, $B^H_{r_i}(u_i) \subseteq B^G_{r_i}(u_i)$.
We then have that $V(G) = V(H) = B^H_{r_1}(u_1) \cup \cdots \cup B^H_{r_k}(u_k) \subseteq B^G_{r_1}(u_1) \cup \cdots \cup B^G_{r_k}(u_k)$, so $\{r_1, \cdots, r_k\}$ covers G.

□

Theorem 2. *Let $0 \leq r_1 \leq r_2$. Then $\{r_1, r_2\}$ is a cover $\iff 2r_1 \leq r_2$.*

We will not prove this theorem in this paper since it has already been proven. However, we will use it many times through our proofs.

Corollary 1. *Let $0 \leq 2r_1 \leq r_2$. Then $\{r_1, r_2\}$ covers any graph H with $|H| \leq 2(r_1 + r_2) + 2$.*

Proof. Let H be a graph with $|H| \leq 2(r_1 + r_2) + 2$. Let $k = 2(r_1 + r_2) - 2 - |H|$ and P_k a path of length k.
Construct G by choosing some vertices $u \in H, v \in P_5$ and setting $V(G) = V(H) \cup V(P_5)$ and $E(G) = E(P_5) \cup E(H) \cup \{uv\}$.
G is connected and $|G| = |P_k| + |H| = 2(r_1 + r_2) + 2$. By **Theorem 4**, $\{r_1, r_2\}$ cover G, so by **Proposition 4**, $\{r_1, r_2\}$ covers H'.

□

Definition 4. *Let T be a tree. We say that $T' \subseteq T$ is a corner-subtree of T if $T \backslash T'$ is connected.*

Lemma 1. *Let T be a tree and $T' \subset T$ a proper subtree. Then T' is a corner-subtree of $T \iff T \backslash T'$ is a corner-subtree of T.*

Proof. We claim that $T' = T\backslash(T\backslash T')$. To see this, let $u \in T$ and notice that $u \in T' \iff u \notin T\backslash T' \iff u \in T\backslash(T\backslash T')$, so $V(T') = V(T\backslash(T\backslash T'))$.
Now, if $E(T) = \emptyset$, then T is a single vertex and has no proper subtree, so $E(T) \neq \emptyset$.
Let now $uv \in E(T)$ and notice that $uv \in E(T') \iff u, v \in T' \iff u, v \notin T\backslash T' \iff u, v \in T\backslash(T\backslash T') \iff uv \in E(T\backslash(T\backslash T'))$, so $E(T') = E(T\backslash(T\backslash T'))$.
We have that $T' = T\backslash(T\backslash T')$, the claim is proven.

Since T' is a tree, that means $T\backslash(T\backslash T')$ is connected.
If T' is a corner-subtree of T, then $T\backslash T'$ is connected, hence also a corner-subtree of T since $T\backslash(T\backslash T')$, proving ($\implies$).

For the other direction, if $T\backslash T'$ is a corner-subtree of T, then it is connected, and by using the claim, we have that T' is connected, hence T' is a corner-subtree, proving ($\impliedby$).

□

Lemma 2. *Let $0 \leq r$ and G a graph with longest path of size less or equal to $2r + 1$. Then r covers G.*

Proof. Let $P \subseteq G$ be the longest path in G. Let $T \subseteq G$ be a spanning tree for G such that $P \subseteq T$. We have that $V(G) = V(T)$ and $E(T) \subseteq E(G)$.
Let $v_1, v_2 \in T$ such that $P = P^T_{v_1,v_2}$. We know v_1, v_2 exist since T is a tree. Let $a \in T$ such that $P \subseteq B^T_r(a)$. a must exist since $|P| \leq 2r + 1$.
Let $u \in T$.
Let $i \in \{1, 2\}$ such that $a \in P^T_{v_i,u}$, i exists, otherwise $P^T_{v_1,v_2} \cup P^T_{v_2,u} \cup P^T_{u,v_1}$ would contain a cycle and T wouldn't be a tree. Let $i \neq j \in \{1, 2\}$.
We then have that $P_{v_i,u} = P_{v_i,a} \cup P_{a,u}$ and that $d(v_i, a) + d(a, u) = d(v_i, u) \leq d(v_i, v_j) = d(v_i, a) + d(a, v_j) \implies d(a, u) \leq d(a, v_j) \leq r$, hence $u \in B_r(a)$.
Since u was chosen arbitrarily, $T \subseteq B_r(a)$. By **Proposition 3**, r covers G.

□

Corollary 2. *Let $0 \leq r$, T a tree and $P \subseteq G$ its longest path with end vertex v. Assume that $\{r\}$ doesn't cover G, then there is a vertex $a \in P$ such that $d(a, v) = r$.*

Proof. Assume not, let $u \neq v$ be the other end vertex of the path, then $|P| - 1 = d(u, v) < r \implies |P| \leq r$, so by **Lemma 2**, $\{r\}$ covers T, a contradiction.

□

Lemma 3. *Let $r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T a tree of size $2(r_1+r_2+r_3)+3$ such that $\{r_1, r_2, r_3\}$ is not a cover for T. Let $u \in V(T)$ and assume that $\{r_1, r_2, r_3\}$ doesn't cover T. If $|B_{r_i}(u)| \geq 2r_i + 1$ for $i \in \{1, 2\}$, then $B_{r_i}(u)$ is not a corner-subtree of T.*

Proof. Assume that $B_{r_i}(u)$ is a corner-subtree of T.
Let $i \neq j \in \{1, 2\}$. Set $T' = T \backslash B_{r_i}(u)$. Then $|T'| = |T| - |B_{r_i}(u)| \leq 2(r_j + r_3) + 2$. Since T' is connected, by **Corollary 1**, we can pick $u_j, u_3 \in T'$ such that $V(T') = B_{r_j}(u_j) \cup B_{r_3}(u_3)$. Then $V(T) = V(T') \cup B_{r_i}(u_i) = B_{r_j}(u_1) \cup B_{r_i}(u_i) \cup B_{r_3}(u_3)$. This contradicts the fact that $\{r_1, r_2, r_3\}$ is not a cover for T, so $B_{r_i}(u)$ cannot be a corner-subtree of T.

□

Lemma 4. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. Assume that $\{r_1, r_2, r_3\}$ doesn't cover T, then $|l_i| \leq 2r_1$ for all $i \in \{1, 2, 4, 6, 7\}$.*

Proof. Assume that $|l_i| > 2r_1$ for some $i \in \{1, 2, 4, 6, 7\}$, so $|l_i| \geq 2r_1 + 1$. Pick $a \in l_i$ such that $d(v_i, a) = r_1$, then $B_{r_1}(a) \subseteq l_1$, $T \backslash (B_{r_1}(a))$ is connected and $|B_{r_1}(a)| = 2r_1 + 1$. By **Lemma 2**, $B_{r_1}(a)$ is not a corner-subtree of T, but since $T \backslash (B_{r_1}(a))$ is connected, we have a contradiction.

□

Lemma 5. *Let $0 \leq r$ and T a tree with longest path P with $|P| \geq r_3 + 1$. If v is an end-vertex of P and $a \in P$ is such that $d(a, v) = r$, then $B_{2r_3}(v) \subseteq B_{r_3}(a)$.*

Proof. Let v be an end-vertex of P and $a \in P$ such that $d(a, v) = r$. Let $u \in B_{2r_3}(v)$.
If $a \notin P_{v,u}$, then since P is the longest path, we must have that $d(a, u) \leq d(a, v) = r_3$, hence $u \in B_{r_3}(a)$.
If $a \in P_{v,u}$, then $d(v, u) = d(v, a) + d(a, u)$ since T is a tree $\implies$ $2r_3 \leq$

$r_3 + d(a,u) \implies d(a,u) \leq r_3 \implies u \in B_{r_3}(a)$.
Since u was chosen arbitrarily in $B_{2r_3}(a)$, we have that $B_{2r_3}(v) \subseteq B_{r_3}(a)$.

□

Lemma 6. *Let $0 \leq r_1, r_2$, T a tree and $P \subseteq T$ a subpath with end-vertex v and $r_1 \leq |P| \leq 2r_1 + 2r_2 + 2$. Let $a \in P$ such that $d(a,v) = r_1$, then r_2 covers $P \backslash B_{r_1}(a)$.*

Proof. $|P \cap B_{r_1}(a)| = 2r_1 + 1$, so $|P \backslash B_{r_1}(a)| \leq 2r_2 + 1$, so by **Lemma 2**, r_2 covers $P \backslash B_{r_1}(a)$.

□

Lemma 7. *Let $0 \leq r_1, \cdots, r_k$ and P be a path with $2(r_1 + \cdots + r_k) + k$ vertices. Then $\{r_1, \cdots, r_k\}$ covers P.*

Proof. We argue by induction on k.
The base case $k = 1$ is trivial by **Lemma 2**.

Now, assume the lemma is true for $k \in \mathbb{N}$. Let $0 \leq r_1, \cdots, r_{k+1}$ and $P = \{u_1, \cdots, u_{(2r_1+\cdots+2r_{k+1}+k+1)}\}$ be a path of size $2r_1 + \cdots + 2r_{k+1} + k + 1$.
By induction hypothesis, we know we can cover the subpath $P' = \{u_1, \cdots, u_{(2r_1+\cdots+2r_k+k)}\}$ with $r_1, \cdots, r_k$. Then $P \backslash P'$ is a path of length $2r_{k+1} + 1$, so we can cover it with r_{k+1} by **Lemma 6**. P is covered by $r_1, \cdots, r_{k+1}$.

□

Now that we have all these results, we can start focusing on **Theorem 1**. We will prove it through many distinct propositions, since it takes a lot of space to write. Each of the following propositions will treat a different case, and all of the different cases will have been proven at the end.

Proposition 4. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. If P_{v_1,v_2} is the longest path in T, then $\{r_1, r_2, r_3\}$ covers T.*

Proof. Assume that $\{r_1, r_2, r_3\}$ doesn't cover T. By **Lemma 3**, $|l_1|, |l_2| \leq 2r_1 \leq 2r_2 \leq r_3$, so $d(u_{123}, v_1), d(u_{123}, v_2) \leq r_3$. Assume now that there is some $u \in$

$T \backslash P_{v_1,v_2}$ such that $d(u, u_{123}) > d(v_2, u_{123})$. Then P_{v_1,v_2} is a shorter path than $P_{v_1,u}$, which contradicts the fact that P_{v_1,v_2} is the longest path in T, so $d(u, u_{123}) \leq d(v_2, u_{123})$ for all $u \in T \backslash P_{v_1,v_2}$. So $T = P_{v_1,v_2} \cup (T \backslash P_{v_1,v_2}) \subseteq B_{r_3}(u_{123})$, so $\{r_3\}$ covers T, so by **Proposition 2**, $\{r_1, r_2, r_3\}$ covers T.

□

Proposition 5. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. If P_{v_1,v_4} is the longest path in T, then $\{r_1, r_2, r_3\}$ covers T.*

Proof. Assume that $\{r_1, r_2, r_3\}$ doesn't cover T. By symmetry, we may assume that $|l_7| \geq |l_6|$. We separate the proof into two cases.

Case 1: $|P_{v_1,v_4}| \leq 2r_3 + 2r_2 + 2$.
Choose $c \in P_{v_1,v_4}$ such that $d(c, v_4) = r_3$. By **Lemma 4**, $|l_4| \leq 2r_2 \leq r_3$, so $c \in l_{1,3} \cup \{u_{345}\}$. Since P_{v_1,v_4} is the longest path in T, $d(c, v_4) \geq d(c, v_7) \geq d(c, v_6)$, so $l_{5,6,7} \subseteq B_{r_3}(c)$.
Now, by **Lemma 4**, we can choose $a \in l_2$ such that $l_2 \subseteq B_{r_1}(a)$.
Finally, by **Lemma 6**, we can choose $b \in P \backslash B_{r_3}(c)$ such that $B_{r_2}(b) \subseteq P \backslash B_{r_3}(c)$.
We get that $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1, r_2, r_3\}$ covers T.
Case 1 leads to a contradiction.

Case 2: $|P_{v_1,v_4}| > 2r_3 + 2r_2 + 2$.
We will prove a series of inequalities.

$$|P_{v_4,v_7}| \geq 2r_2 + 2 : \qquad (1)$$

Assume the inverse inequality. Since (1) is false, we can choose $b \in P_{v_4,v_7}$ such that $l_{4,5,7} \subseteq B_{r_2}(b)$, and since $|l_6| \leq |l_7|, |l_{4,5}|$, then $l_6 \subseteq B_{r_2}(b)$.
Now, if $l_3 \nsubseteq B_{r_2}(b)$, that would give that $B_{r_2}(b)$ is a corner-subtree with $|B_{r_2}(b)| \geq |P_{v_1,v_4} \cap B_{r_2}(b)| = 2r_2 + 1$, so by **Lemma 3**, we get a contradiction, so $l_3 \subseteq B_{r_2}(b)$.
Now, by **Lemma 4**, we can choose $a \in l_1, c \in l_2$ such that $l_1 \subseteq B_{r_1}(a)$ and $l_2 \subseteq B_{r_1}(c) \subseteq B_{r_3}(c)$.
We finally get that $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1, r_2, r_3\}$ covers T, this is a contradiction, which means (1) is proven.

$$|P_{v_1,v_2}| \geq 2r_2 + 2 : \qquad (2)$$

Assume the inverse inequality. Since (2) is false, we can choose $b \in P_{v_1,v_2}$ such that $l_{1,2} \subseteq B_{r_2}(b)$.

Now, if $l_3 \nsubseteq B_{r_2}(b)$, that would give that $B_{r_2}(b)$ is a corner-subtree with $|B_{r_2}(b)| \geq |P_{v_1,v_4} \cap B_{r_2}(b)| = 2r_2+1$, so by **Lemma 3**, we get a contradiction, so $l_3 \subseteq B_{r_2}(b)$. Furthermore, by **Lemma 4**, $d(u_{345}, v_6) \leq d(u_{345}, v_7) \leq d(u_{345}, v_4) = |l_4| \leq 2r_1 \leq r_3$, so $l_{4,5,6,7} \subseteq B_{r_3}(u_{345})$.
We obtain that $T = B_{r_2}(b) \cup B_{r_3}(u_{345})$, so $\{r_2, r_3\}$ covers T, so by **Proposition 2**, $\{r_1, r_2, r_3\}$ covers T, a contradiction. We have proven (2).

$$|P_{v_2,v_4}| \geq 2r_2 + 2r_3 + 3 : \tag{3}$$

Assume the inverse inequality. Choose $c \in P_{v_2,v_4}$ such that $d(c, v_4) = r_3$. Similarly to Case 1 and 2, $l_{4,5,6,7} \subseteq B_{r_3}(c)$.
Now, by **Lemma 6**, we can choose $b \in P_{v_2,v_4} \backslash B_{r_3}(c)$ such that $P_{v_2,v_4} \backslash B_{r_3}(c) \subseteq B_{r_2}(b)$.
Finally, by **Lemma 4**, we can choose $a \in l_1$ such that $l_1 \subseteq B_{r_1}(a)$.
We get that $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1, r_2, r_3\}$ covers T, a contradiction. We have proven (3).

$$|P_{v_1,v_7}| \geq 2r_2 + 2r_3 + 3 : \tag{4}$$

Assume the inverse inequality. Choose $c \in P_{v_1,v_7}$ such that $d(c, v_7) = r_3$. Similarly to Case 1, 2, and 3, $l_{4,5,6,7} \subseteq B_{r_3}(c)$.
Now, by **Lemma 6**, we can choose $b \in P_{v_1,v_7} \backslash B_{r_3}(c)$ such that $P_{v_1,v_7} \backslash B_{r_3}(c) \subseteq B_{r_2}(b)$.
Finally, by **Lemma 4**, we can choose $a \in l_2$ such that $l_2 \subseteq B_{r_1}(a)$.
We get that $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1, r_2, r_3\}$ covers T, a contradiction. We have proven (4).

Now, lets add (1),(2),(3), and (4):
$|P_{v_4,v_7}| + |P_{v_1,v_2}| + |P_{v_2,v_4}| + |P_{v_1,v_7}| \geq (2r_2 + 2) + (2r_2 + 2) + (2r_2 + 2r_3 + 3) + (2r_2 + 2r_3 + 3)$
$\implies (|l_4| + 1 + |l_5| + 1 + |l_7|) + (|l_1| + 1| + |l_2|) + (|l_2| + 1 + |l_3| + 1 + |l_4|) + (|l_1| + 1 + |l_3| + 1 + |l_5| + 1 + |l_7|) \geq 8r_2 + 4r_3 + 10$
$\implies 2|T| > 2(|l_1| + |l_2| + |l_3| + |l_4| + |l_5| + |l_7|) \geq 2(2r_3 + 2r_2 + 2r_1 + 3) + 4$, this is impossible.
Case 2 leads to a contradiction.

Both cases lead to a contradiction, which means our initial assumption that $\{r_1, r_2, r_3\}$ doesn't cover T was false, so $\{r_1, r_2, r_3\}$ covers T.

□

Proposition 6. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1+r_2+r_3)+3$ vertices. If P_{v_1,v_7} is the longest path in T and $\{r_1,r_2,r_3\}$ doesn't cover T, then $|P_{v_1,v_2}|, |P_{v_6,v_7}| \geq 2r_2+2$.*

Proof. We will only prove that $|P_{v_6,v_7}| \geq 2r_2+2$. $|P_{v_1,v_2}| \geq 2r_2+2$ is symmetrically identical.
Assume that $|P_{v_6,v_7}| \leq 2r_2+1$. First notice that by **Lemma 3**, we must have that $|l_6| \geq 1$.
We will now prove a series of inequalities.

$$|P_{v_1,v_4}| \geq 2r_1+2r_3+3 : \tag{5}$$

Assume the inverse inequality. Choose $a \in P_{v_1,v_7}$ such that $d(a,v_7)=r_2$, a exists by **Corollary 2**. Then $l_{6,7} \subseteq B_{r_2}(b)$ since $|l_7| \geq |l_6|$ and by (5), hence $l_{5,6,7} \subseteq B_{r_2}(b)$, otherwise $B_{r_2}(b)$ would be a corner-subtree of size at least $2r_2+1$, which contradicts **Lemma 3**.
Now, choose $c \in P_{v_1,v_4}$ with $d(c,v_1)=r_3$, so $l_{1,2} \subseteq B_{r_3}(c)$ by **Lemma 5** and we can choose $a \in P_{v_1,v_4} \backslash B_{r_3}(c)$ such that $P_{v_1,v_4} \backslash B_{r_3}(c) \subseteq B_{r_1}(a)$ by **Lemma 6**. Hence we would have that $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1,r_2,r_3\}$ covers T, a contradiction. (5) is proven.

$$|P_{v_2,v_7}| \geq 2r_1+2r_3+3 : \tag{6}$$

Assume the inverse inequality. $|l_4|+1 \leq 2r_1+1$ and $|l_5|+1+|l_7| \leq 2r_2$, so $|l_4|+1+|l_5|+1+|l_7| \leq 4r_2+1 \leq 2r_3+1$. Thus, we can choose $c \in P_{v_1,v_7}$ such that $d(c,v_7)=r_3$ by **Corollary 2**, so $l_{4,5,6,7} \subseteq B_{r_3}(c)$.
We can then choose $a \in P_{v_1,v_7} \backslash B_{r_3}(c)$ such that $P_{v_1,v_7} \backslash B_{r_3}(c) \subseteq B_{r_1}(a)$ and $b \in l_2$ such that $l_2 \subseteq B_{r_2}(b)$. Then $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1,r_2,r_3\}$ covers T, a contradiction.

$$|P_{v_1,v_2}| \geq 2r_1+2 : \tag{7}$$

Assume the inverse inequality. By **Lemma 4**, we can choose $c \in l_4$ such that $l_4 \subseteq B_{r_3}(c)$. Next, by **Corollary 2**, we can choose $a \in P_{v_1,v_4}$ such that $d(a,v_1)=r_1$ and $b \in P_{v_4,v_7}$ such that $d(b,v_7)=r_2$, then similarly to the proof of (5), $l_{5,6,7} \subseteq B_{r_2}(b)$ and $l_{1,2,3} \subseteq B_{r_1}(a)$, so $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, so $\{r_1,r_2,r_3\}$ covers T, a contradiction. (7) is proven.

$$|P_{v_4,v_7}| \geq 2r_1+2 : \tag{8}$$

Assume the inverse inequality. By **Lemma 2** applied on $l_{4,5,6,7}$, we can choose $b \in l_{4,5,6,7}$ such that $l_{4,5,6,7} \subseteq B_{r_2}(b)$, but then $l_3 \subseteq B_{r_2}(b)$, otherwise $B_{r_2}(b)$ would be a corner-subtree, which would contradict **Lemma 3**. We can then cover l_1 with r_3 and l_2 with r_1 by **Lemma 4**, $\{r_1, r_2, r_3\}$ would cover T, which is a contradiction.

Knowing $|l_6| \geq 1$, we can add inequalities (5), (6), (7), and (8) to get that $2|T| + 3 \geq 2|l_1| + 2|l_2| + 2|l_3| + 2|l_4| + 2|l_5| + 2|l_7| + 8 = |P_{v_1,v_4}| + |P_{v_2,v_7}| + |P_{v_1,v_2}| + |P_{v_4,v_7}| \geq 4r_3 + 4r_2 + 4r_1 + 10 \geq 2|T| + 4$, a contradiction.
We get that $|l_6| + 1 + |l_7| > 2r_2 + 1$, so $|l_6| + |l_7| \geq 2r_2 + 1$.

□

Proposition 7. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. If P_{v_1,v_7} is the longest path in T, $|l_{2,3}| \leq |l_{5,6}|$,, and $\{r_1, r_2, r_3\}$ doesn't cover T, then $|P_{v_1,v_4}| \geq 2r_3 + 2$.*

Proof. Assume that $|P_{v_1,v_4}| \leq 2r_3$. We will prove a series of inequalities.

$$|P_{v_1,v_6}| \geq 2r_2 + 2r_3 + 3 : \tag{9}$$

Assume the inverse inequality. We can choose $c \in P_{v_1,v_7}$ such that $d(c, v_1) = r_3$, and since $|l_4| \leq |l_1| + 1 + |l_3|$ and $|P_{v_1,v_4}| \leq 2r_3$, then $l_{1,2,3,4} \subseteq B_{r_3}(c)$. We can then cover l_7 with r_1 by **Lemma 4** and cover $P_{v_1,v_6} \backslash B_{r_3}(c)$ with r_2, so $\{r_1, r_2, r_3\}$ covers T, a contradiction. (9) is proven.

$$|P_{v_4,v_7}| \geq 2r_3 + 2 : \tag{10}$$

Assume the inverse inequality. Then similarly to (9) by symmetry, $|P_{v_2,v_7}| \geq 2r_2 + 2r_3 + 2$. Also, from **Proposition 6**, $|P_{v_1,v_2}|, |P_{v_6,v_7}| \geq 2r_2 + 1$.
Adding all of this together with (9), we get that $2|T| - 2|l_4| + 2 = 2|l_1| + 2|l_2| + 2|l_3| + 2|l_5| + 2|l_6| + 2|l_7| + 8 = |P_{v_1,v_6}| + |P_{v_2,v_7}| + |P_{v_1,v_2}| + |P_{v_6,v_7}| \geq 8r_2 + 4r_3 + 8 \geq 2|T| + 2$, which implies $|l_4| = 0$.
That means $v_4 = u_{345}$, so $|T| + 1 \geq |P_{v_1,u_{345}}| + |P_{u_{345},v_7}| \geq |P_{v_1,v_4}| + |P_{v_4,v_7}| \geq 4r_3 + 4 \geq 2r_3 + 2r_2 + 2r_1 + 4 = |T| + 1$
$\implies |T| = |P_{v_1,v_7}|$, so T is a path, which contradicts **Lemma 7**. (10) is proven.

$$|P_{v_2,v_4}| \geq 2r_2 + 2 : \tag{11}$$

Assume the inverse inequality. We study two cases.

Case 1: $|l_4| \geq |l_{2,3}|$
We have that $|P_{v_4,v_7}| \geq 2r_2 + 2r_3 + 3$. Otherwise, we could choose $b \in P_{v_4,v_7}$ such that $d(b, v_4) = r_2$, then $l_{2,3,4} \subseteq B_{r_2}(b)$ by our case and we can choose $c \in P_{v_4,v_7} \backslash B_{r_2}(b)$ such that $l_{4,5,6,7} \backslash B_{r_2}(b)$ by **Lemma 6** and because $|l_6| \leq |l_7|$. Then we could choose $a \in l_1$ such that $l_1 \subseteq B_{r_1}(a)$. Thus $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, a contradiction.
From this, (9) and **Proposition 6** ($|P_{v_6,v_7}|, |P_{v_1,v_2}| \geq 2r_2 + 2$), we can add these inequalities together to get $2|T| - |l_2| - |l_3| - |l_4| = 2|l_1| + |l_2| + |l_3| + |l_4| + 2|l_5| + 2|l_6| + 2|l_7| + 6 = |P_{v_4,v_7}| + |P_{v_1,v_6}| + |P_{v_6,v_7}| \geq 8r_2 + 4r_3 + 10 \geq 2|T| + 4 \implies 0 \geq |l_1| + |l_2| + |l_3| + |l_4| + 4 \geq 4$, contradiction.

Case 2: $|l_4| < |l_{2,3}|$
We have that $|P_{v_2,v_7}| \geq 2r_2 + 2r_3 + 3$. Otherwise, we could choose $b \in P_{v_2,v_7}$ such that $d(b, v_2) = r_2$, then $l_{2,3,4} \subseteq B_{r_2}(b)$ by our case and we can choose $c \in P_{v_4,v_7} \backslash B_{r_2}(b)$ such that $l_{2,3,5,6,7} \backslash B_{r_2}(b)$ by **Lemma 6** and because $|l_6| \leq |l_7|$. Then we could choose $a \in l_1$ such that $l_1 \subseteq B_{r_1}(a)$. Thus $T = B_{r_1}(a) \cup B_{r_2}(b) \cup B_{r_3}(c)$, a contradiction.
From this, (9) and **Proposition 6** ($|P_{v_6,v_7}|, |P_{v_1,v_2}| \geq 2r_1 + 2$), we can add these inequalities together to get $2|T| - 2|l_4| + 1 = 2|l_1| + 2|l_2| + 2|l_3| + 2|l_5| + 2|l_6| + 2|l_7| + 7 = |P_{v_1,v_2}| + |P_{v_1,v_6}| + |P_{v_2,v_7}| + |P_{v_6,v_7}| \geq 4r_1 + 4r_2 + 4r_3 + 10 = 2|T| + 6 \implies 0 \leq 5 + 2|l_4| \leq 4$, contradiction.

In both cases, we have reached a contradiction, so (11) is proven.

We can then add (9), (10), (11), and **Proposition 6** ($|P_{v_6,v_7}|, |P_{v_1,v_2}| \geq 2r_1 + 2$). We get that $2|T| + 3 = 2|l_1| + 2|l_2| + 2|l_3| + 2|l_4| + 2|l_5| + 2|l_6| + 2|l_7| + 9 = |P_{v_1,v_6}| + |P_{v_2,v_4}| + |P_{v_4,v_7}| + |P_{v_6,v_7}| + |P_{v_1,v_2}| \geq 4r_1 + 4r_2 + 4r_3 + 11 \geq 2|T| + 5 \implies 3 \geq 5$, a contradiction. We have proven the proposition.

□

Proposition 8. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. If P_{v_1,v_7} is the longest path in T, $|l_{2,3}| \leq |l_{5,6}|$, and $\{r_1, r_2, r_3\}$ doesn't cover T, then $|P_{v_4,v_6}| \geq 2r_3 + 2$.*

Proof. Assume for a contradiction that $|l_4| \leq |l_{2,3}|$.
We will prove a series of inequalities.

$$|P_{v_2,v_6}| \geq 2r_3 + 1 : \tag{12}$$

Assume the inverse inequality. Since $|l_4| \leq |l_{2,3}| \leq |l_{5,6}|$, P_{v_2,v_6} is the longest path in $l_{2,3,4,5,6}$, so by **Lemma 2**, we can cover $l_{2,3,4,5,6}$ with r_3. Then by **Lemma 4**, we can cover l_1 with r_1 and l_7 with r_2. $\{r_1, r_2, r_3\}$ covers T, a contradiction. (12) is proven.

$$|P_{v_4,v_7}| \leq 2r_3 + 1 : \tag{13}$$

Assume the inverse inequality. We can add it with (12), **Proposition 7** and **Proposition 6** ($|P_{v_6,v_7}|, |P_{v_1,v_2}| \geq 2r_2+1$), we get that $2|T|+3 = 2|l_1|+2|l_2|+2|l_3|+2|l_4|+2|l_5|+2|l_6|+2|l_7|+9 = |P_{v_4,v_7}|+|P_{v_2,v_6}|+|P_{v_1,v_4}|+|P_{v_1,v_2}|+|P_{v_6,v_7}| \geq 4r_2+6r_3+10 \geq 4r_1+4r_2+4r_3+10 \geq 2|T|+4$, a contradiction. (13) is proven.

$$|P_{v_2,v_7}| \geq 2r_3 + 2r_2 + 2 : \tag{14}$$

Assume the inverse inequality. Choose $c \in P_{v_1,v_7}$ such that $d(c, v_7) = r_3$, c exists by **Corollary 2**. Then by (13) and because $|l_6| \leq |l_7|$, we have that $l_{4,5,6,7} \subseteq B_{r_3}(c)$. We can then cover $P_{v_2,v_7} \backslash B_{r_3}(c)$ with r_2 by **Lemma 6** and l_1 by **Lemma 4**. $\{r_1, r_2, r_3\}$ covers T, a contradiction. (14) is proven.

$$|P_{v_4,v_6}| \geq 2r_1 + 2 : \tag{15}$$

Assume the inverse inequality. Then r_1 covers $l_{4,5,6}$.
Furthermore, $|l_{1,3}| = |l_1|+1+|l_3| \leq |l_1|+|l_2|+1+|l_3| = |l_1|+|l_{2,3}| \leq |l_1|+|l_{5,6}| \leq |l_1| + |P_{v_4,v_6}| \leq 2r_1 + 2r_1 + 1 \leq 2r_3 + 1$ by **Lemma 4**.
So we can cover $l_{1,3}$ with r_3. But then since $|l_{1,2}| = |l_1|+1+|l_2| \leq 2r_1+1+2r_1 \leq 2r_3 + 1$ by **Lemma 4**, then r_3 can cover $l_{1,2,3}$.
Finally, by **Lemma 4**, r_2 can cover l_7.
$\{r_1, r_2, r_3\}$ can cover T, a contradiction. (15) is proven.

Now, we can add **Proposition 6** ($|P_{v_1,v_2}| \geq 2r_1, |P_{v_6,v_7}| \geq 2r_2+2$), **Proposition 7**, (14), and (15). We get that $2|T|+3 = 2|l_1|+2|l_2|+2|l_3|+2|l_4|+2|l_5|+2|l_6|+2|l_7|+9 = |P_{v_1,v_2}|+|P_{v_6,v_7}|+|P_{v_1,v_4}|+|P_{v_2,v_7}|+|P_{v_4,v_6}| \geq 4r_1+4r_2+4r_3+10 = 2|T|+4$, a contradiction.
So $|l_4| > |l_{2,3}|$.

Now, assume that $|P_{v_2,v_6}| \leq 2r_3 + 1$.
Since $|l_4| > |l_{2,3}|$, P_{v_2,v_6} is the longest path in $l_{2,3,4,5,6}$. Then, by **Lemma 2**, we can cover $l_{2,3,4,5,6}$ with r_3. Then, by **Lemma 4**, we can cover l_1 with r_1 and l_7 with r_2. So $\{r_1, r_2.r_3\}$ covers T, a contradiction.
That means $|P_{v_2,v_6}| \geq 2r_3 + 2$. The proposition is proven. □

Proposition 9. *Let $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and T be a 5-leaves tree as in Figure 1 that has $2(r_1 + r_2 + r_3) + 3$ vertices. If P_{v_1,v_7} is the longest path in T, $|l_{2,3}| \leq |l_{5,6}|$, and $\{r_1, r_2, r_3\}$ doesn't cover T, then $|P_{v_2,v_7}| \geq 2r_3 + 2$.*

Proof. Assume for a contradiction that $|P_{v_2,v_7}| \leq 2r_3 + 1$.
Choose $c \in P_{v_2,v_7}$ such that $d(c, v_7) = r_3$. c exists by **Corollary 2**. Since $|l_6| \leq |l_7|$ and $|P_{v_2,v_7}| \leq 2r_3 + 1$, then $l_{2,3,5,6,7} \subseteq B_{r_3}(c)$. Then, by **Lemma 4**, we can cover l_4 with r_2 and l_1 with r_1. $\{r_1, r_2, r_3\}$ covers T, a contradiction.
We have proven the proposition.

□

We now introduce a new type of subgraph. Let T be a tree. Let L be the set of leaves of T. We know $L \neq \emptyset$, so $|T \backslash L| < |T|$.
If we keep repeatedly removing the set of leaves, we must eventually reach an empty graph, which has 0 leaf.
Let's take the last step before obtaining the empty graph. This tree only contains leafs, so only degree ≤ 1 vertices. The only possible such trees are P_1 and P_2, which have 1 and 2 leafs respectively. That means there must be a step during the process of removing leaves where we had a tree with at most 5 leaves.
We introduce this notation:

Notation 7. *The first graph with at most k leaves obtained by repeating the process described above starting from tree T is called the fundamental k-subtree of T.*

We will then need one last result before proving **Theorem 1**:

Proposition 10. *Let $1 \leq r_1, \cdots, r_k$, T be a tree with at least 3 vertices and L be its set of leaves. If $\{r_1, \cdots, r_k\}$ doesn't cover T, then $\{r_1 - 1, \cdots, r_k - 1\}$ doesn't cover $T \backslash L$.*

Proof. We will prove the contrapositive. Assume $\{r_1 - 1, r_2 - 1, r_3 - 1\}$ covers $T \backslash L \subseteq T$. Let $u_1, \cdots, u_k \in T \backslash L$ such that $V(T \backslash L) = B_{r_1 - 1}(u_1) \cup \cdots \cup B_{r_k - 1}(u_k)$.
Notice that $B_{r_i - 1}(u_i) \subseteq B_{r_i}(u_i)$ for any $1 \leq i \leq k$.
Let now $v \in T$. If $v \in T \backslash L$, then choose $1 \leq i \leq k$ such that $v \in B_{r_i - 1}(u_i)$, then $v \in B_{r_i}(u_i)$.

If $v \notin T\backslash L$, then $v \in L$. Take $v \neq u \in N(v)$. u is not a leaf of T, so $u \in T\backslash L$. Choose $1 \leq i \leq k$ such that $u \in B_{r_i-1}(u_i)$. Then $d(v, u_i) = |P_{u_i,v}| - 1 = |P_{u_i,u}| = d(u_i, u) + 1 \leq r_i$, hence $v \in B_{r_i}(u_i)$.
Since v was chosen arbitrarily in T, then $T = B_{r_1}(u_1) \cup \cdots \cup B_{r_k}(u_k)$.
$\{r_1, \cdots, r_k\}$ covers T, the contrapositive is proven.

□

Corollary 3. *Let $1 \leq r_1, \cdots, r_k$ and T be a tree with at least 3 vertices. Let H be the fundamental $(2k-1)$-subtree of T and let m be the number of leaves deletions needed to go from T to H. If $\{r_1, \cdots, r_k\}$ doesn't cover T, then $\{r_1 - m, \cdots, r_k - m\}$ doesn't H.*

Proof. First notice that $\{r_1 - 1, \cdots, r_k - 1\}$ doesn't cover $T\backslash L$ where L is the set of leaves of T by **Proposition 10**. Then $\{r_1 - 2, \cdots, r_k - 2\}$ doesn't cover $T\backslash L\backslash L'$ where L' is the set of leaves of $T\backslash L$. This reasoning can be repeated until we reach H. Then $\{r_1 - m, \cdots, r_k - m\}$ doesn't cover H.

□

We now have all the preliminary results we need to prove **Theorem 1**:

Theorem 1. *If $0 \leq r_1 \leq r_2 \leq r_3$ and $2r_2 \leq r_3$, then $\{r_1, r_2, r_3\}$ is a cover.*

Proof. Assume this theorem is false for a contradiction. This means there exist $0 \leq r_1 \leq r_2 \leq r_3$ with $2r_2 \leq r_3$ and a connected graph G with $2(r_1+r_2+r_3)+3$ vertices such that $\{r_1, r_2, r_3\}$ doesn't cover G.

Let $H \subseteq G$ be a spanning tree of G. Then by **Proposition 3**, $\{r_1, r_2, r_3\}$ doesn't cover H.

Let now T be the fundamental 5-subtree of H, $|T| \geq 3$. Let m be the number of leaves deletions needed to go from T to H. By **Corollary 3**, since $\{r_1, r_2, r_3\}$ doesn't cover H and $5 = 2 * 3 - 1$, then $\{r_1 - m, r_2 - m, r_3 - m\}$ doesn't cover T.

Now, at each leaves deletion we made to go from H to T, we know we removed at least 6 leaves. Otherwise, T wouldn't be the fundamental 5-subtree of H. So $|T| \leq |H| - 6m = |G| - 6m = 2(r_1 + r_2 + r_3) + 3 - 6m = 2((r_1 - m) + (r_2 - m) + (r_3 - m)) + 3$.

Set $r'_i = r_i - m$ for each $i \in \{1, 2, 3\}$.
$\cdot r_1 \leq r_2 \leq r_3 \implies r_1 - m \leq r_2 - m \leq r_3 - m \implies r'_1 \leq r'_2 \leq r'_3$.
$\cdot 2r_2 \leq r_3 \implies 2r'_2 = 2r_2 - 2m \leq r_3 - 2m \leq r_3 - m = r'_3 \implies 2r'_2 \leq r'_3$.
$\cdot |T| \leq 2((r_1 - m) + (r_2 - m) + (r_3 - m)) + 3 = 2r'_1 + 2r'_2 + 2r'_3 + 3$.
$\cdot \{r_1 - m, r_2 - m, r_3 - m\}$ doesn't cover $T \implies \{r'_1, r'_2, r'_3\}$ doesn't cover T.

Let T' be a copy of T to which we have attached one path with $2(r'_1 + r'_2 + r'_3) + 3 - |T|$ vertices to one of its leaves. T' is still a 5-leaves tree and by **Proposition 1**, $\{r'_1, r'_2, r'_3\}$ doesn't cover T'.

So we have $0 \leq r'_1 \leq r'_2 \leq r'_3$ with $2r'_1 \leq r'_3$ and T' a 5-leaves tree with $2(r'_1 + r'_2 + r'_3) + 3$ vertices such that $\{r'_1, r'_2, r'_3\}$ doesn't cover T'. T' can be described as in *Figure 1*.

By symmetry, we may assume that the longest path in T' contains v_1.
We now have three cases:

Case 1: P_{v_1,v_2} is the longest path in T'.
By **Proposition 4**, $\{r'_1, r'_2, r'_3\}$ covers T', a contradiction.

Case 2: P_{v_1,v_4} is the longest path in T'.
By **Proposition 5**, $\{r'_1, r'_2, r'_3\}$ covers T', a contradiction.

Case 3: P_{v_1,v_7} is the longest path in T'.
By adding all the results from **Propositions 6,7,8,9**, we get that $2|T| + 3 = 2|l_1| + 2|l_2| + 2|l_3| + 2|l_4| + 2|l_5| + 2|l_6| + 2|l_7| + 9 = |P_{v_1,v_2}| + |P_{v_6,v_7}| + |P_{v_1,v_4}| + |P_{v_4,v_6}| + |P_{v_2,v_7}| \geq 4r'_2 + 6r'_3 + 10 \geq 4r'_1 + 4r'_2 + 4r'_3 + 10 = 2|T'| + 4$, a contradiction.

Case 4: P_{v_1,v_6} is the longest path in T'.
This case is the same as Case 3 by symmetry, so we still have a contradiction.

In all cases, we reach a contradiction. The theorem is proven.

□