SUBELLIPTIC OPERATORS

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FINAL REPORT

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Contents

1 Introduction

This report consists in a summary and highlights of three months of research on subelliptic operators. The first goal of this project was first to study the Grushin 3D-operator, which we will see can be simplified to a 2D-operator, and find its eigenfunctions. These eigenfunctions must also satisfy the Dirichlet conditions, which means the image of those functions must be equal to 0 at $x = 1$ and $x = -1$.
Depending on what we obtain, we might or might not need to use approximations of eigenfunctions. In the case where we unfortunately need approximations, this might decrease the number of eigenfunctions we will be able to find. When we find functions, we will graph them on Desmos so that we can see what they look like on $x \in [-1, 1]$.

The second goal of this project was to find specific sets of couples $(l, m) \in \mathbb{Z}^2$ and study linear combinations of Legendre functions associated to these couples. We would then try to graph and observe the zero sets of all those linear combinations.

2 Important clarification before starting

For simplicity, in this document, we will consider that the 0-mapping function is an eigenfunction.

We will use the term non-zero eigenfunction to refer to an eigenfunction that is surely not the 0-mapping function.

3 Facts and discoveries that will help us

3.1 Simplifying the Grushin operator

We want to find all eigenfunctions (defined on $x \in [-1, 1]$) of the Grushin operator $\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}$ of the form $e^{iky}V(x)$ where $k \in \mathbb{N}$.
We also need that $V(-1) = V(1) = 0$.

Fact: $e^{iky}V(x)$ is an eigenfunction of $\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \iff V(x)$ is an eigenfunction of $-\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$.

Proof:
($\implies$)
Let $e^{iky}V(x)$ be an eigenfunction of $\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}$, then $(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2})(e^{iky}V(x)) = \alpha e^{iky}V(x)$ for some $\alpha \in \mathbb{R}$
$\implies \alpha e^{iky}V(x) = e^{iky}\frac{d^2V}{dx^2} + x^2(\frac{d^2}{dy^2}e^{iky})V(x) = e^{iky}\frac{d^2V}{dx^2} - k^2x^2e^{iky}V(x)$
$\implies (-\frac{\alpha}{k^2})V(x) = -\frac{1}{k^2}\frac{d^2V}{dx^2} + x^2V(x)$
$\implies V(x)$ is an eigenfunction of $-\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$.

($\impliedby$)
Let $V(x)$ be an eigenfunction of $-\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$. Then $-\frac{1}{k^2}\frac{d^2V}{dx^2} + x^2V(x) = \alpha V(x)$ for some $\alpha \in \mathbb{R}$.
$(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2})(e^{iky}V(x)) = e^{iky}\frac{d^2V}{dx^2} + x^2\frac{d^2}{dy^2}(e^{iky})V(x) = e^{iky}\frac{\partial^2V}{\partial x^2} - k^2x^{eiky}V(x) = -k^2e^{iky}(-\frac{1}{k^2}\frac{d^2V}{dx^2} + x^2V(x)) = -k^2e^{iky}\alpha V(x) = (-\alpha k^2)(e^{iky}V(x))$
$\implies e^{iky}V(x)$ is an eigenfunction of $\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}$.

Q.E.D.

Thanks to this fact, we can simplify our goal by trying to find all eigenfunctions $V(x)$ of $-\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ instead, where $V(1) = V(-1) = 0$.

To do so, we will use the power series method (see next section).

3.2 Power series method

Assume $V(x) = \sum_{n=0}^{\infty} c_n x^n$ is an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$. Then for some $E \in \mathbb{R}$,

$\frac{E}{k^2}V(x) = -\frac{1}{k^2}\frac{d^2}{dx^2}V(x) + x^2V(x)$

$\implies \frac{E}{k^2}\sum_{n=0}^{\infty} c_n x^n = -\frac{1}{k^2}\frac{d^2}{dx^2}(\sum_{n=0}^{\infty} c_n x^n) + x^2\sum_{n=0}^{\infty} c_n x^n$

$= \sum_{n=0}^{\infty} -\frac{1}{k^2}\frac{d^2}{dx^2}(c_n x^n) + \sum_{n=0}^{\infty} c_n x^{n+2}$

$= \sum_{n=0}^{\infty} -\frac{1}{k^2}n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$

$\implies \sum_{n=0}^{\infty} \frac{E}{k^2}c_n x^n = \sum_{n=0}^{\infty} -\frac{1}{k^2}n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$

$\implies 0 = \sum_{n=0}^{\infty} \frac{E}{k^2}c_n x^n + \sum_{n=0}^{\infty} \frac{1}{k^2}n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+2}$

$\implies 0 = \sum_{n=0}^{\infty} \frac{E}{k^2}c_n x^n + \sum_{n=2}^{\infty} \frac{1}{k^2}n(n-1)c_n x^{n-2} + \frac{1}{k^2} * 0 * (0-1)c_0 x^{0-2}$
$+ \frac{1}{k^2} * 1 * (1-1)c_1 x^{1-2} - \sum_{n=0}^{\infty} c_n x^{n+2}$

$\implies 0 = \sum_{n=0}^{\infty} \frac{E}{k^2}c_n x^n + \sum_{n=2}^{\infty} \frac{1}{k^2}n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+2}$

$\implies 0 = \sum_{n=0}^{\infty} Ec_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} k^2 c_{n-2}x^n$

$\implies 0 = \sum_{n=2}^{\infty} Ec_n x^n + Ec_0x^0 + Ec_1x^1 + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n$
$+ (0+2)(0+1)c_{0+2}x^0 + (1+2)(1+1)c_{1+2}x^1 - \sum_{n=2}^{\infty} k^2 c_{n-2}x^n$

$\implies 0 = \sum_{n=2}^{\infty} Ec_n x^n + (Ec_0 + 2c_2) + (Ec_1 + 6c_3)x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n$
$- \sum_{n=2}^{\infty} k^2 c_{n-2}x^n$

$\implies 0 = \sum_{n=2}^{\infty}[Ec_n + (n+2)(n+1)c_{n+2} - k^2c_{n-2}]x^n + (Ec_0 + 2c_2) + (Ec_1 +$
$6c_3)x$

$\implies Ec_0 + 2c_2 = 0$ and $Ec_1 + 6c_3 = 0$ and
$Ec_n + (n+2)(n+1)c_{n+2} - k^2c_{n-2} = 0$ for all $n \geq 2$

$\implies c_2 = -\frac{E}{2}c_0$ and $c_3 = -\frac{E}{6}c_1$ and $c_n = \frac{k^2c_{n-4} - Ec_{n-2}}{n(n-1)}$ for all $n \geq 4$

$\frac{E}{k^2}$ is the eigenvalue of our eigenfunction $V(x)$.

We have now obtained our power series with arbitrary c_0 and c_1. However, we notice that the c_n's are defined by c_{n-2} and c_{n-4} for all $n \geq 4$. This is a situation for which we have not found a way to obtain an exact function. We gladly invite the readers to find one. We will then have to approximate it.

3.3 Verifying our power series makes sense

We know that $e^{\frac{x^2}{2}}$ is an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ if $k = 1$ and $E = -1$

Proof:
$\hat{H}(e^{\frac{x^2}{2}}) = -\frac{d^2}{dx^2}e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}} = -\frac{d}{dx}(xe^{\frac{x^2}{2}}) + x^2e^{\frac{x^2}{2}} =$
$-(e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}}) + x^2e^{\frac{x^2}{2}} = -1 * e^{\frac{x^2}{2}}$.

Now that we know $V(x) = e^{\frac{x^2}{2}}$ is an eigenfunction of $\hat{H}$, let's look at its taylor expansion around $x = 0$.

$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \implies e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{x^2}{2})^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

First, we have $c_1 = c_3 = ... = 0$.

Next, we have $c_0 = \frac{1}{2^0 * 0!} = 1$ and $c_2x^2 = \frac{x^{2*1}}{2^1 1!} = \frac{x^2}{2} \implies c_2 = \frac{1}{2} = -\frac{-1}{2} = -\frac{E}{2}$.

Also, we have $c_{2n} = \frac{1}{2^n n!}$, thus $c_{2n+2} = c_{2(n+1)} = \frac{1}{2^{n+1}(n+1)!}$.

Finally, $\frac{k^2c_{2n} - Ec_{2n+2}}{(2n+4)(2n+3)} = \frac{1^2c_{2n} - (-1)c_{2n+2}}{(2n+4)(2n+3)} = \frac{c_{2n} + c_{2n+2}}{(2n+4)(2n+3)} = \frac{\frac{1}{2^n n!} + \frac{1}{2^{n+1}(n+1)!}}{(2n+4)(2n+3)} =$

$\frac{\frac{2(n+1)}{2^{n+1}(n+1)!} + \frac{1}{2^{n+1}(n+1)!}}{(2n+4)(2n+3)} = \frac{2(n+1)+1}{2^{n+1}(n+1)!(2n+4)(2n+3)} = \frac{2n+3}{2^{n+1}(n+1)!2(n+2)(2n+3)} = \frac{1}{2^{n+2}(n+1)!(n+2)} =$

$\frac{1}{2^{n+2}(n+2)!} = c_{2(n+2)} = c_{2n+4}$.

So $c_{2n+4} = \frac{k^2c_{2n} - Ec_{2n+2}}{(2n+4)(2n+3)}$.

With all of that, we can see that $e^{\frac{x^2}{2}}$ perfectly fits the power series. This is a good evidence that our power series makes sense.

3.4 Quick study of our power series

We will prove some facts that will simplify our work in order to find eigenfunctions/eigenvalues.

First, we can notice that our power series can be separated into two subseries: one with the odd terms and one with the even terms. Let's verify that.

Let $N_e = \{n \in \mathbb{N}_0 : n \text{ is even}\} = \{0, 2, 4, 6, 8...\}$,
$N_o = \{n \in \mathbb{N}_0 : n \text{ is odd}\} = \{1, 3, 5, 7, 9, ...\}$.

Let $N_{e\geq 4} = \{n \in \mathbb{N} : n \geq 4 \text{ and } n \text{ is even}\}$ and
$N_{o\geq 4} = \{n \in \mathbb{N} : n \geq 4 \text{ and } n \text{ is odd}\}$.

$V(x) = [c_0 - \frac{E}{2}c_0x^2 + \sum_{n\in N_{e\geq 4}} c_nx^n] + [c_1x - \frac{E}{6}c_1x^3 + \sum_{n\in N_{o\geq 4}} c_nx^n]$

Fact: For all $n \in \mathbb{N}_e$, $c_n \propto c_0$.
Proof:
$c_0 \propto c_0$ and $c_2 = -\frac{E}{2}c_0 \propto c_0$ by definition.
Now, assume $c_n \propto c_0$ and $c_{n+2} \propto c_0$. Then $c_{n+4} = \frac{k^2c_n - Ec_{n+2}}{(n+4)(n+3)} \propto c_0$.
By induction, we have proven the fact.

Very similarly, we can prove that for all $n \in \mathbb{N}_o$, $c_n \propto c_1$.

Furthermore, we can prove very similarly that for all $n \in \mathbb{N}_e$, c_n is independent of c_1 and for all $n \in \mathbb{N}_o$, c_n is independent of c_0.

With all that said, we can establish two subfunctions
$V_e(x) = \sum_{n\in N_e} c_nx^n = c_0 - \frac{E}{2}c_0x^2 + \sum_{n\in N_{e\geq 4}} c_nx^n$ and
$V_o(x) = \sum_{n\in N_o} c_nx^n = c_1x - \frac{E}{6}c_1x^3 + \sum_{n\in N_{o\geq 4}} c_nx^n$
where $V(x) = V_e(x) + V_o(x)$.

So if $c_0 = 0$, then $V(x) = V_o(x)$ and if $c_1 = 0$, then $V(x) = V_e(x)$.

We now know that $V(x)$ can be separated into two subfunctions, but the most important facts that will simplify our work a lot are the following two.

Fact: $V_e(x)$ and $V_o(x)$ are eigenfunctions of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalues $\frac{E}{k^2}$.

Proof: If we set $c_1 = 0$, we get that $V(x) = V_e(x)$, hence $V_e(x)$ is an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue $\frac{E}{k^2}$.
If we set $c_0 = 0$, we get that $V(x) = V_o(x)$, hence $V_o(x)$ is an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue $\frac{E}{k^2}$.

Fact: $V_e(1) = V_e(-1) = V_o(1) = V_o(-1) = 0$.

Proof:
$V_e(1) = V_e(-1) = \sum_{n \in N_e} c_n$ and $V_o(1) = -V_o(-1) = \sum_{n \in N_o} c_n$.

$V(1) = V_e(1) + V_o(1) = 0 = V(-1) = V_e(-1) + V_o(-1)$
$\implies V_e(1) + V_o(1) = V_e(1) - V_o(1)$
$\implies 2V_o(1) = 0 \implies V_o(1) = 0$
$\implies V_e(1) = V_e(-1) = V_o(1) = V_o(-1) = 0.$

With all this information acquired, we can conclude a crucial fact:

If $V(x)$ is an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue $\frac{E}{k^2}$ and $V(1) = V(-1) = 0$, then $V_e(x)$ and $V_o(x)$ both have those same exact properties.
i.e. $V_e(x)$ and $V_o(x)$ are eigenfunctions of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalues $\frac{E}{k^2}$ and $V_e(1) = V_e(-1) = V_o(1) = V_o(-1) = 0$

Let's now assume that $P_e(x) = \sum_{n \in \mathbb{N}_e}^{\infty} c_n x^n$ (only even terms) is one existing NONZERO eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue α and that $P_e(1) = P_e(-1) = 0$.

Now, if there exists a nonzero eigenfunction $P_o(x) = \sum_{n \in \mathbb{N}_o}^{\infty} c_n x^n$ (only odd terms) of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue α and $P_o(1) = P_o(-1) = 0$, then we know that for any $a, b \in \mathbb{R}$, $aP_e(x) + bP_o(x)$ is an eigenfunction with eigenvalue α and that any eigenfunction with eigenvalue α can be written as a linear combination of $P_e(x)$ and $P_o(x)$.

However, if such a function $P_o(x)$ doesn't exist, then the only eigenfunction of odd terms existing is $P_o(x) = 0$, which means that the only nonzero eigenfunctions with eigenvalue α that exist are of the form $aP_e(x) + bP_o(x) = aP_e(x)$ where $a \in \mathbb{R}$ and that any function of the form $aP_e(x)$ is an eigenfunction with eigenvalue α.

The same principle applies if $P_o(x) = \sum_{n\in\mathbb{N}_o}^{\infty} c_n x^n$ (only odd terms) is one existing nonzero eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue α and $P_o(1) = P_o(-1) = 0$.

Also, any function $P(x) = aP_e(x) + bP_o(x)$ has the property that $P(1) = aP_e(1) + bP_o(1) = a * 0 + b * 0 = 0$ and that $P(-1) = aP_e(-1) + bP_o(-1) = a * 0 + b * 0 = 0$.

So let's say we have found a function $P_e(x)$ with eigenvalue α. We can then seek for a function $P_o(x)$ with eigenvalue α.

If we find one, then we have generated two degrees of freedom of eigenfunctions of the form $aP_e(x) + bP_o(x)$ where $a, b \in \mathbb{R}$ and those are all the existing functions with eigenvalue α.

If we don't find one, then all the eigenfunctions with eigenvalue α are of the form $aP_e(x)$ where $a \in \mathbb{R}$.

3.5 Extra interesting fact that will not help us

Extra fact 1: Let $\psi(x) : \mathbb{R} \to \mathbb{R}$ be continous. $\psi(x)$ and $x\psi(x)$ are eigenfunctions of $\hat{H} = -\frac{d^2}{dx^2} + x^2 \iff \psi(x) = Ae^{\pm\frac{x^2}{2}}$ for some $A \in \mathbb{R}$.

i.e. $\psi(x) = Ae^{\pm\frac{x^2}{2}}$ are the only eigenfunctions of $\hat{H}$ that have the property that multiplying them by x gives us another eigenfunction.

Proof:
Let $\psi(x)$ be continous.

($\Longrightarrow$) Assume $\psi(x) \neq 0$ and $x\psi(x)$ are eigenfunctions of $\hat{H} = -\frac{d^2}{dx^2} + x^2$.
Then $-\frac{d^2\psi}{dx^2} + x^2\psi = \alpha_1\psi$ for some $\alpha_1 \in \mathbb{R}^*$.
Also, $-\frac{d^2}{dx^2}(x\psi(x)) + x^3\psi = \alpha_2 x\psi$ for some $\alpha_2 \in \mathbb{R}^*$
$\Longrightarrow -\frac{d}{dx}(\psi(x) + x\frac{d\psi}{dx}) + x^3\psi = \alpha_2 x\psi$
$\Longrightarrow -\frac{d\psi}{dx} - \frac{d\psi}{dx} - x\frac{d^2\psi}{dx^2} + x^3\psi = \alpha_2 x\psi$
$\Longrightarrow -2\frac{d\psi}{dx} + x[-\frac{d^2\psi}{dx^2} + x^2\psi] = \alpha_2 x\psi$
$\Longrightarrow -2\frac{d\psi}{dx} + x\alpha_1\psi = \alpha_2 x\psi$
$\Longrightarrow \frac{d\psi}{dx} = \frac{\alpha_1-\alpha_2}{2}x\psi = 2bx\psi$ where $b \in \mathbb{R}$
$\Longrightarrow \int \frac{d\psi}{\psi} = b\int 2x dx$
$\Longrightarrow ln|\psi| = bx^2 + ln|A|$ for some $A \in \mathbb{R}^*$
$\Longrightarrow |\psi(x)| = e^{bx^2+ln|A|} = |A|e^{bx^2}$
$\Longrightarrow \psi(x) = Ae^{bx^2}$.

Now suppose that Ae^{bx^2} is an eigenfunction of $\hat{H}$, then $\hat{H}(\psi) = \alpha\psi$ for some $\alpha \in \mathbb{R}$
$\Longrightarrow -\frac{d^2\psi}{dx^2} + x^2\psi = \alpha\psi$
$\Longrightarrow -\frac{d}{dx}[2Abxe^{bx^2}] + Ax^2e^{bx^2} = -2Ab[e^{bx^2} + 2bx^2e^{bx^2}] + Ax^2e^{bx^2}$
$= -2Abe^{bx^2} + A[1-4b^2]x^2e^{bx^2} = \alpha Ae^{bx^2}$
$\Longrightarrow 1 - 4b^2 = 0$
$\Longrightarrow b = \pm\frac{1}{2} \Longrightarrow \psi(x) = Ae^{\pm\frac{x^2}{2}}$.

($\Longleftarrow$) Let $\psi(x) = Ae^{\pm\frac{x^2}{2}}$ for some $A \in \mathbb{R}^*$. Then $\hat{H}(\psi) =$
$-\frac{d^2\psi}{dx^2} + x^2\psi = -\frac{d}{dx}[\pm Axe^{\pm\frac{x^2}{2}}] + Ax^2e^{\pm\frac{x^2}{2}} =$
$-A[\pm e^{\pm\frac{x^2}{2}} + x^2e^{\pm x^2}] + Ax^2e^{\pm x^2} = \mp Ae^{\pm\frac{x^2}{2}}$
$\Longrightarrow \psi$ is an eigenfunction of $\hat{H}$.

Finally, $\hat{H}(x\psi) = -\frac{d^2(x\psi)}{dx^2} + x^2(x\psi) = -\frac{d}{dx}[Ae^{\pm\frac{x^2}{2}} + x\frac{d\psi}{dx}] + x^3\psi =$
$\mp Axe^{\pm\frac{x^2}{2}} - [\frac{d\psi}{dx} + x\frac{d^2\psi}{dx^2}] + x^3\psi = \mp Axe^{\pm\frac{x^2}{2}} \mp Axe^{\pm\frac{x^2}{2}} + x[-\frac{d^2\psi}{dx^2} + x^2\psi] =$
$\mp 2x\psi + x[\mp\psi] = \mp 3x\psi$
$\Longrightarrow x\psi(x)$ is an eigenfunction of $\hat{H}$.

4 Finding the wanted eigenfunctions

4.1 Methodology

The method we will use to find eigenfunctions goes as follows:

0. Define the function $T(E, k, c_0, c_1, x)$ as an eigenfunction of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$ with eigenvalue $\frac{E}{k^2}$ and c_0, c_1 as the first two coefficients of the power series describing it.

1. We will set $c_0 = 1$ and $c_1 = 0$ and fix a wanted value for $k \in \mathbb{N}$ (let's say $k = 1$ for now).

2. We will only use a finite number of terms of our power series (let's say we use up to the m'th degree term, depending on how precise we want to be and on the power of our computer). Let's write it $T_m(E, 1, 1, 0, x) = \sum_{n=0}^{m} c_n x^n$.

3. We will replace x by 1 and m by 14 to get a fairly good approximation. We get $T_{14}(E, 1, 1, 0, 1)$, this value should be equal to 0, since our function must equal 0 for $x = -1$ and $x = 1$. We now have $T_{14}(E, 1, 1, 0, 1)$, which is only dependent on E.

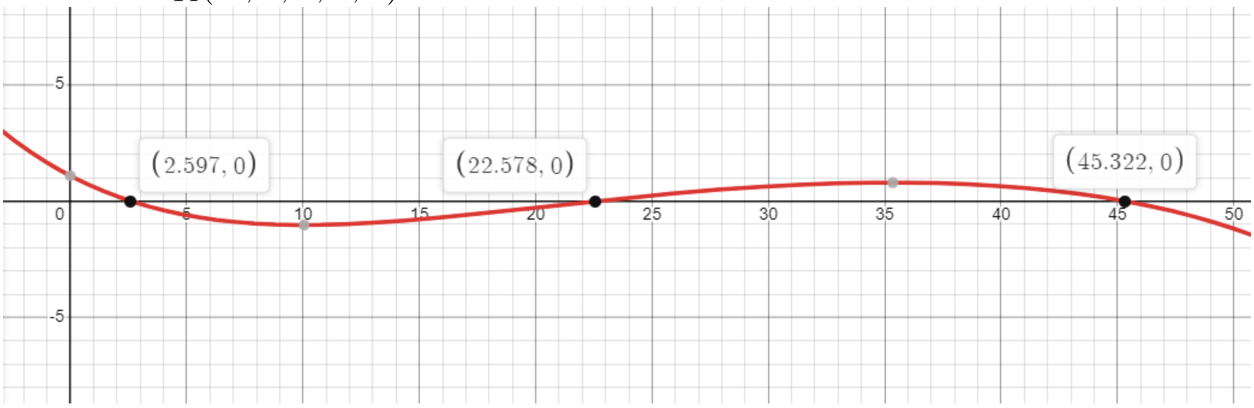
4. We will finally find for which values of E do we have $T_{14}(E, 1, 1, 0, 1) = 0$. There are many ways to do it, we will do it by graphing $T_{14}(E, 1, 1, 0, 1)$ on Desmos with E being the independent variable and by looking at the zeros (see next page).

5. We will redo the same process but by setting $c_0 = 0$ and $c_1 = 1$ instead.

4.2 Finding one eigenfunction by graphing

Graph for $c_0 = 1$, $c_1 = 0$, $k = 1$, $m = 14$ and $x = 1$:

We have $T_{14}(E, 1, 1, 0, 1)$:



We see that $T_{14}(E, 1, 1, 0, 1) = 0$ at $E = 2.597$, $E = 22.578$ and $E = 45.322$.

So we have that $T_{14}(2.597, 1, 1, 0, x)$ is (approximately) an eigenfunction of $\hat{H} = -\frac{d^2}{dx^2} + x^2$ with eigenvalue $E = 2.597$.

If we develop and simplify $T_{14}(2.597, 1, 1, 0, x)$, we get
$T_{14}(2.597, 1, 1, 0, x) = -0.000007542x^{14} + 0.0000976x^{12} - 0.001119x^{10} + 0.009976x^8 - 0.07482x^6 + 0.3644x^4 - 1.2985x^2 + 1$

We will verify that $T_{14}(2.597, 1, 1, 0, x)$ satisfies the conditions it needs to satisfy.

If we apply $-\frac{d^2}{dx^2} + x^2 - 2.597$ on $T_{14}(2.597, 1, 1, 0, x)$, we should get a result that is very close to 0 for any $x \in [-1, 1]$. Let's try that:

$(-\frac{d^2}{dx^2} + x^2 - 2.597)(T_{14}(2.597, 1, 1, 0, x)) =$

$-(-0.001373x^{12} + 0.01288x^{10} - 0.1007x^8 + 0.5587x^6 - 2.245x^4 + 4.373x^2 - 2.597) +$
$(x^2 - 2.597)(-0.000007542x^{14} + 0.0000976x^{12} - 0.001119x^{10} + 0.009976x^8 - 0.07482x^6 + 0.3644x^4 - 1.2985x^2 + 1)$

$= -0.000007542x^{16} + 0.0001172x^{14}$

The coefficients we obtained are very close to 0, which is a very good clue that $T_{14}(2.597, 1, 1, 0, x)$ is indeed an eigenfunction of $-\frac{d^2}{dx^2} + x^2$ with eigenvalue $E = 2.597$.

Finally, if we evaluate $T_{14}(2.597, 1, 1, 0, x)$ at $x = -1$ and $x = 1$, we get $T_{14}(2.597, 1, 1, 0, -1) = T_{14}(2.597, 1, 1, 0, 1) = -0.0005502$, which is very close to 0. So our eigenfunction indeed approximately satisfies all the needed conditions.

4.3 Quickly finding the two remaining eigenfunctions

According to our graph, we still have two eigenfunctions that we can find with eigenvalues 22.597 and 45.322.
We will use the exact same procedure to find the functions:

$T_{14}(22.578, 1, 1, 0, x) = -0.05839x^{14} + 0.3848x^{12} - 1.9340x^{10} + 7.0028x^8 - 16.4244x^6 + 21.324x^4 - 11.289x^2 + 1$

$T_{14}(45.322, 1, 1, 0, x) = -5.231x^{14} + 19.7875x^{12} - 55.2726x^{10} + 106.8874x^8 - 130.1801x^6 + 85.6706x^4 - 22.661x^2 + 1$

Let's now apply $\hat{H} - E$ on each function.

$(-\frac{d^2}{dx^2} + x^2 - 22.578)T_{14}(22.578, 1, 1, 0, x) = -0.05839x^{16} + 1.703x^{14}$

$(-\frac{d^2}{dx^2} + x^2 - 45.322)T_{14}(45.322, 1, 1, 0, x) = -5.231x^{16} + 256.88x^{14}$

As we can see, the higher the eigenvalue is, the less accurate the approximation of the eigenfunction seems to be. However, before concluding anything, we will use the method of projections in order to see if indeed the approximation gets worse as E increases.

4.4 Method of projections

Let $M_0(P) = \int_{-1}^{1} P^2 dx$
$M_1(P,k,E) = \int_{-1}^{1} ([-\frac{1}{k^2}\frac{d^2}{dx^2} + x^2 - E]P)^2 dx$.

If $(\frac{M_1(P,k,E)}{M_0(P)})^{\frac{1}{2}}$ is close to 0, then our approximation is good. Let's first check that for $T_{14}(2.597,1,1,0,x)$:

$M_0(T_{14}(2.597,1,1,0,x)) = \int_{-1}^{1}(-0.000007542x^{14}+0.0000976x^{12}-0.001119x^{10}+ 0.009976x^8 - 0.07482x^6 + 0.3644x^4 - 1.2985x^2 + 1)^2 dx = 0.9857$

$M_1(T_{14}(2.597,1,1,0,x),1,2.597) = \int_{-1}^{1}(-0.000007542x^{16}+0.0001172x^{14})^2 dx$
$= 8.36692\text{x}10^{-10}$

$(\frac{M_1(T_{14}(2.597,1,1,0,x),1,2.597)}{M_0(T_{14}(2.597,1,1,0,x))})^{\frac{1}{2}} = (\frac{8.36692E-10}{0.9857})^{\frac{1}{2}} = 0.0000291347$

$M_0(T_{14}(22.578,1,1,0,x)) = \int_{-1}^{1}(-0.05839x^{14} + 0.3848x^{12} - 1.9340x^{10} + 7.0028x^8 - 16.4244x^6 + 21.324x^4 - 11.289x^2 + 1)^2 dx = 1.00453$

$M_1(T_{14}(22.578,1,1,0,x),1,22.578) = \int_{-1}^{1}(-0.05839x^{16} + 1.703x^{14})^2 dx$
$= 0.18739$

$(\frac{M_1(T_{14}(22.578,1,1,0,x),1,22.578)}{M_0(T_{14}(22.578,1,1,0,x))})^{\frac{1}{2}} = (\frac{0.18739}{1.00453})^{\frac{1}{2}} = 0.4319$

$M_0(T_{14}(45.322,1,1,0,x)) = \int_{-1}^{1}(-5.231x^{14} + 19.7875x^{12} - 55.2726x^{10} + 106.8874x^8 - 130.1801x^6 + 85.6706x^4 - 22.661x^2 + 1)^2 dx = 0.9060$

$M_1(T_{14}(45.322,1,1,0,x),1,45.322) = \int_{-1}^{1}(-5.231x^{16} + 256.88x^{14})^2 dx$
$= 4379.12$

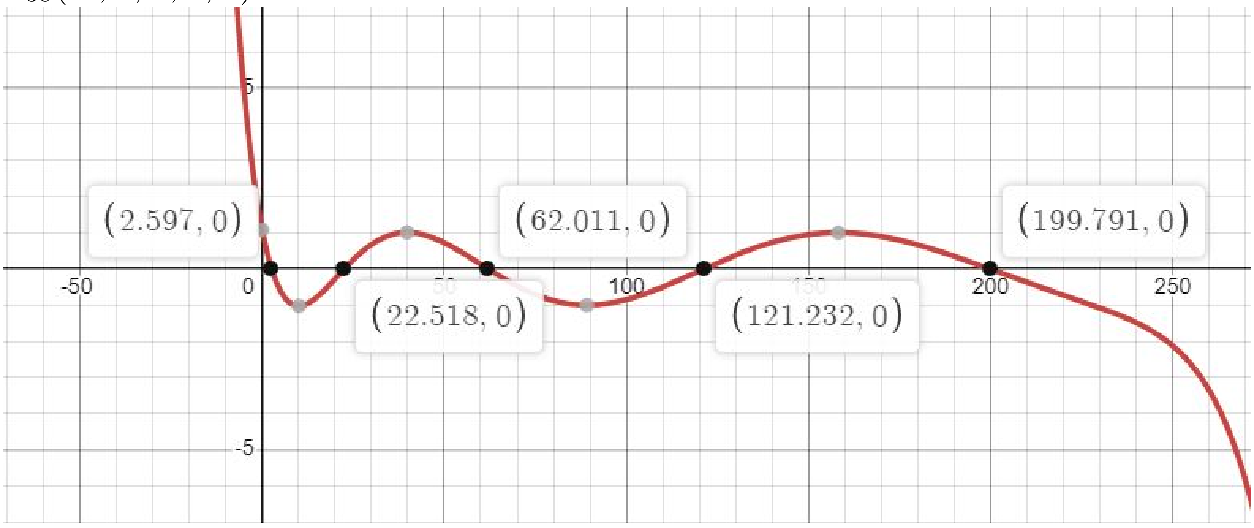
$(\frac{M_1(T_{14}(45.322,1,1,0,x),1,45.322)}{M_0(T_{14}(45.322,1,1,0,x))})^{\frac{1}{2}} = (\frac{4379.12}{0.9060})^{\frac{1}{2}} = 69.52$

As we can see, the estimation indeed becomes less accurate as E increases. To solve this problem, we will simply add more terms to our power series so that their accuracy is satisfying.

4.5 Finding more accurate eigenvalues

Reminder: we have to remember that we estimated our 3 values of E, which means they might not be accurate. For this reason, we will restart the process from the beginning, but with 38 terms to find the eigenvalues.

$T_{38}(E, 1, 1, 0, 1)$:



We notice here that our first eigenvalue is exactly the same (which is coherent with the fact that its estimation was extremely good). Also, the second eigenvalue has slightly changed and the third one has changed a lot.

We also notice .that we now have two new eigenvalues: $E = 121.232$ and $E = 199.791$. Following what we have seen before, we can suspect that those eigenvalues are not accurate at all, but that if we increase the value of m for $T_m(E, 1, 1, 0, 1)$, those eigenvalues will get more and more accurate.

We will now find which functions are associated with $E = 22.518$, $E = 62.011$, $E = 121.232$ and $E = 199.791$, starting with $m = 14$ for $T_m(E, 1, 1, 0, x)$.

4.6 Finding more accurate eigenfunctions

Let's start with $E = 22.518$ and $E = 62.011$ and verify their accuracy:

$T_{14}(22.518, 1, 1, 0, x) = -0.05745x^{14} + 0.3793x^{12} - 1.9153x^{10} + 6.931x^8 - 16.2962x^6 + 21.2108x^4 - 11.259x^2 + 1$

$T_{14}(62.011, 1, 1, 0, x) = -43.8819x^{14} + 124.6107x^{12} - 259.2713x^{10} + 370.935x^8 - 332.393x^6 + 160.307x^4 - 31.0055x^2 + 1$

$(-\frac{d^2}{dx^2}+x^2-22.518)T_{14}(22.518,1,1,0,x) = -0.05744x^{16}+1.6728x^{14}$

$M_0(T_{14}(22.518,1,1,0,x)) = \int_{-1}^{1}(-0.05745x^{14}+0.3793x^{12}-1.9153x^{10}+6.931x^8-16.2962x^6+21.2108x^4-11.259x^2+1)^2dx = 1.00692$

$M_1(T_{14}(22.518,1,1,0,x),1,22.518) = \int_{-1}^{1}(-0.05744x^{16}+1.6728x^{14})^2dx = 0.18079$

$(\frac{M_1(T_{14}(22.518,1,1,0,x),1,22.518)}{M_0(T_{14}(22.518,1,1,0,x))})^{\frac{1}{2}} = (\frac{0.18079}{1.00692})^{\frac{1}{2}} = 0.4237$

$(-\frac{d^2}{dx^2}+x^2-62.011)T_{14}(62.011,1,1,0,x) = -43.8819x^{16}+2845.77x^{14}$

$M_0(T_{14}(62.011,1,1,0,x)) = \int_{-1}^{1}(-43.8819x^{14}+124.6107x^{12}-259.2713x^{10}+370.935x^8-332.393x^6+160.307x^4-31.0055x^2+1)^2dx = 5.9472$

$M_1(T_{14}(62.011,1,1,0,x),1,62.011) = \int_{-1}^{1}(-43.8819x^{16}+2845.77x^{14})^2dx = 542514$

$(\frac{M_1(T_{14}(62.011,1,1,0,x),1,62.011)}{M_0(T_{14}(62.011,1,1,0,x))})^{\frac{1}{2}} = (\frac{542514}{5.9472})^{\frac{1}{2}} = 302.029$

For $E = 22.718$, our eigenfunction is a bit more accurate than for $E = 22.578$, but still not enough. Also, even though $E = 62.011$ is a way more accurate eigenvalue than $E = 45.322$, the eigenfunction we get out of it is way less accurate (with $m = 14$). To solve both of these problems, we will simply add more terms to our eigenfunctions ($m = 38$):

$T_{38}(22.518,1,1,0,x) = -(9.8139e-17)x^{38}+(2.8057e-15)x^{36}-(7.4804e-14)x^{34}+(1.8507e-12)x^{32}-(4.2255e-11)x^{30}+(8.8447e-10)x^{28}-(1.6845e-8)x^{26}+(2.8933e-7)x^{24}-0.000004434x^{22}+0.00005986x^{20}-0.0007007x^{18}+0.006970x^{16}-0.05745x^{14}+0.3793x^{12}-1.9153x^{10}+6.931x^8-16.2962x^6+21.2108x^4-11.259x^2+1$

$T_{38}(62.011,1,1,0,x) = -(1.1811e-10)x^{38}+(2.1182e-9)x^{36}-(3.4712e-8)x^{34}+(5.1642e-7)x^{32}-0.000006924x^{30}+0.00008295x^{28}-0.0008796x^{26}+0.008164x^{24}-0.06547x^{22}+0.4467x^{20}-2.5463x^{18}+11.8574x^{16}-43.8819x^{14}+124.6107x^{12}-259.2713x^{10}+370.935x^8-332.393x^6+160.307x^4-31.0055x^2+1$

$(-\frac{d^2}{dx^2}+x^2-22.518)T_{38}(22.518,1,1,0,x) = -9.8139776e{-}17x^{40}+5.0156536e{-}15x^{38}$

$M_0(T_{38}(22.518,1,1,0,x)) = \int_{-1}^{1}(-(1.1811e-10)x^{38}+(2.1182e-9)x^{36}-(3.4712e-8)x^{34}+(5.1642e-7)x^{32}-0.000006924x^{30}+0.00008295x^{28}-0.0008796x^{26}+0.008164x^{24}-0.06547x^{22}+0.4467x^{20}-2.5463x^{18}+11.8574x^{16}-43.8819x^{14}+124.6107x^{12}-259.2713x^{10}+370.935x^{8}-332.393x^{6}+160.307x^{4}-31.0055x^{2}+1)^2dx \approx 1.007$

$M_1(T_{38}(22.518,1,1,0,x),1,22.518) = \int_{-1}^{1}((-9.8139e-17)x^{40}+(5.0157e-15)x^{38})^2dx = 6.2875e-31$

$(\frac{M_1(T_{38}(22.518,1,1,0,x),1,22.518)}{M_0(T_{38}(22.518,1,1,0,x))})^{\frac{1}{2}} = (\frac{6.2875e-31}{1.007})^{\frac{1}{2}} = 7.9017e-16$

$(-\frac{d^2}{dx^2}+x^2-62.011)T_{38}(62.011,1,1,0,x) = -(1.1811e{-}10)x^{40}+(9.4424e{-}9)x^{38}$

$M_0(T_{38}(62.011,1,1,0,x)) = \int_{-1}^{1}(-(1.1811e-10)x^{38}+(2.1182e-9)x^{36}-(3.4712e-8)x^{34}+(5.1642e-7)x^{32}-0.000006924x^{30}+0.00008295x^{28}-0.0008796x^{26}+0.008164x^{24}-0.06547x^{22}+0.4467x^{20}-2.5463x^{18}+11.8574x^{16}-43.8819x^{14}+124.6107x^{12}-259.2713x^{10}+370.935x^{8}-332.393x^{6}+160.307x^{4}-31.0055x^{2}+1)^2dx \approx 1.003$

$M_1(T_{38}(62.011,1,1,0,x),1,62.011) = \int_{-1}^{1}(-(1.1811e-10)x^{40}+(9.4424e-9)x^{38})^2dx$
$= 2.2597e-18$

$(\frac{M_1(T_{38}(62.011,1,1,0,x),1,62.011)}{M_0(T_{38}(62.011,1,1,0,x))})^{\frac{1}{2}} = (\frac{2.2597e-18}{1.003})^{\frac{1}{2}} = 1.5010e-9$

Now, we see that our functions are extremely good approximations of eigenfunctions. Now that we have found the eigenfunctions we wanted, we will try the same with the two remaining eigenvalues we have.

4.7 Finding the two remaining eigenfunctions

We want to find the power series associated with $E = 121.232$ and $E = 199.791$ with $m = 38$:

$T_{38}(121.232, 1, 1, 0, x) = -0.00001240x^{38} + 0.0001332x^{36} - 0.001292x^{34} + 0.01124x^{32} - 0.08703x^{30} + 0.5952x^{28} - 3.5599x^{26} + 18.4156x^{24} - 81.3568x^{22} + 302.3795x^{20} - 928.7739x^{18} + 2307.09x^{16} - 4511.91x^{14} + 6713.59x^{12} - 7265.57x^{10} + 5373.37x^{8} - 2477.04x^{6} + 612.47x^{4} - 60.616x^{2} + 1$

$T(199.791, 1, 1, 0, x) = -0.1197x^{38} + 0.8174x^{36} - 5.0175x^{34} + 27.5083x^{32} - 133.7228x^{30} + 571.647x^{28} - 2128.8869x^{26} + 6832.7025x^{24} - 18663.988x^{22} + 42754.969x^{20} - 80704.42x^{18} + 122871.5019x^{16} - 146932.306x^{14} + 133408.0928x^{12} - 87943.4397x^{10} + 39560.4876x^{8} - 11080.199x^{6} + 1663.2685x^{4} - 99.8955x^{2} + 1$

Let's now see how accurate these functions are:

$(-\frac{d^2}{dx^2} + x^2 - 121.232)T_{38}(121.232, 1, 1, 0, x) = -0.00001240x^{40} + 0.001637x^{38}$

$M_0(T_{38}(121.232, 1, 1, 0, x)) = \int_{-1}^{1}(-0.00001240x^{38} + 0.0001332x^{36} - 0.001292x^{34} + 0.01124x^{32} - 0.08703x^{30} + 0.5952x^{28} - 3.5599x^{26} + 18.4156x^{24} - 81.3568x^{22} + 302.3795x^{20} - 928.7739x^{18} + 2307.09x^{16} - 4511.91x^{14} + 6713.59x^{12} - 7265.57x^{10} + 5373.37x^{8} - 2477.04x^{6} + 612.47x^{4} - 60.616x^{2} + 1)^2 dx \approx 1.11$

$M_1(T_{38}(121.232, 1, 1, 0, x), 1, 121.232) = \int_{-1}^{1}(-0.00001240x^{40} + 0.001637x^{38})^2 dx = 6.85804e - 8$

$(\frac{M_1(T_{38}(121.232,1,1,0,x),1,121.232)}{M_0(T_{38}(121.232,1,1,0,x))})^{\frac{1}{2}} = (\frac{6.85804e-8}{1.11})^{\frac{1}{2}} = 0.0002486$

Now, all our eigenfunctions are very accurate, except the one with eigenvalue $E = 199.791$. If we really want to solve this problem, we can add more terms to our power series to:

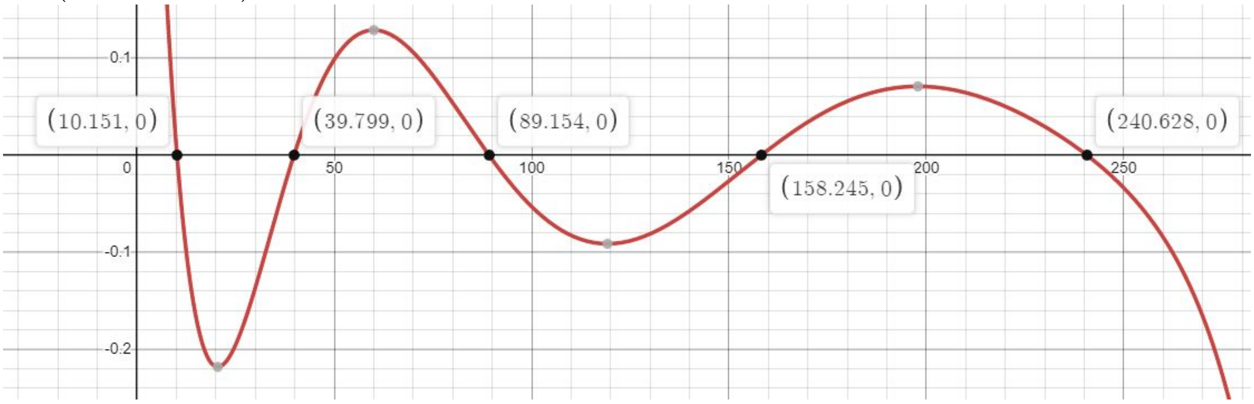
1) Get a more accurate eigenvalue

2) Get a more accurate associated eigenfunction

We will now redo all this process with the odd eigenfunctions.

4.8 Repeat all this process for the odd functions

$T_{39}(E, 1, 0, 1, 1)$:



We have $E = 10.151$, $E = 39.799$, $E = 89.154$, $E = 158.245$ and $E = 240.628$. We will not do it for $E = 240.628$ due to a lack of computational power.

$T_{15}(10.151, 1, 0, 1, x) = -0.000069895x^{15} + 0.0007654x^{13} - 0.006909x^{11} + 0.04926x^9 - 0.2599x^7 + 0.9087x^5 - 1.692x^3 + x$

$T_{23}(39.799, 1, 0, 1, x) = -0.00003658x^{23} + 0.0003802x^{21} - 0.003379x^{19} + 0.02518x^{17} - 0.1535x^{15} + 0.7414x^{13} - 2.7247x^{11} + 7.2115x^9 - 12.7133x^7 + 13.2497x^5 - 6.6332x^3 + x$

$T_{31}(89.154, 1, 0, 1, x) = -0.00003568x^{31} + 0.0003401x^{29} - 0.002862x^{27} + 0.02104x^{25} - 0.1335x^{23} + 0.7211x^{21} - 3.2612x^{19} + 12.1063x^{17} - 36.002x^{15} + 83.1913x^{13} - 143.5971x^{11} + 175.5908x^9 - 141.062x^7 + 66.287x^5 - 14.859 * x^3 + x$

$T_{39}(158.245, 1, 0, 1, x) = -0.00004201x^{39} + 0.0003744x^{37} - 0.003014x^{35} + 0.02178x^{33} - 0.14029x^{31} + 0.7992x^{29} - 3.9917x^{27} + 17.3015x^{25} - 64.3216x^{23} + 202.3247x^{21} - 529.8335x^{19} + 1132.8757x^{17} - 1931.1526x^{15} + 2546.9505x^{13} - 2499.867x^{11} + 1732.746x^9 - 787.064x^7 + 208.729x^5 - 26.374x^3 + x$

We will now measure their accuracies:

$(-\frac{d^2}{dx^2} + x^2 - 10.151)T_{15}(10.151, 1, 0, 1, x) = -0.000069895x^{17} + 0.0083635x^{15}$

$M_0(T_{15}(10.151, 1, 0, 1, x)) = \int_{-1}^{1}(-0.000069895x^{15} + 0.0007654x^{13} - 0.006909x^{11} + 0.04926x^9 - 0.2599x^7 + 0.9087x^5 - 1.692x^3 + x)^2 dx \approx 0.0991042$

$M_1(T_{15}(10.151,1,0,1,x),1,10.151) = \int_{-1}^{1}(-0.000069895x^{17}+0.0083635x^{15})^2 dx$
$= 4.44221e-6$

$(\frac{M_1(T_{15}(10.151,1,0,1,x),1,10.151)}{M_0(T_{15}(199.791,1,0,1,x))})^{\frac{1}{2}} = (\frac{4.44221e-6}{0.0991042})^{\frac{1}{2}} = 0.006695$

$(-\frac{d^2}{dx^2}+x^2-39.799)T_{23}(39.799,1,0,1,x) = -0.00003658x^{25}+0.001836x^{23}$

$M_0(T_{15}(39.799,1,0,1,x)) = \int_{-1}^{1}(-0.00003658x^{23}+0.0003802x^{21}-0.003379x^{19}+0.02518x^{17}-0.1535x^{15}+0.7414x^{13}-2.7247x^{11}+7.2115x^9-12.7133x^7+13.2497x^5-6.6332x^3+x)^2 dx \approx 0.0252167$

$M_1(T_{15}(39.799,1,0,1,x),1,39.799) = \int_{-1}^{1}(-0.00003658x^{25}+0.001836x^{23})^2 dx$
$= 1.38012e-7$

$(\frac{M_1(T_{15}(39.799,1,0,1,x),1,39.799)}{M_0(T_{15}(199.791,1,0,1,x))})^{\frac{1}{2}} = (\frac{1.38012e-7}{0.0252167})^{\frac{1}{2}} = 5.47304e-6$

$(-\frac{d^2}{dx^2}+x^2-89.154)T_{23}(89.154,1,0,1,x) = -0.00003568x^{33}+0.003521x^{31}$

$M_0(T_{15}(89.154,1,0,1,x)) = \int_{-1}^{1}(-0.00003568x^{31}+0.0003401x^{29}-0.002862x^{27}+0.02104x^{25}-0.1335x^{23}+0.7211x^{21}-3.2612x^{19}+12.1063x^{17}-36.002x^{15}+83.1913x^{13}-143.5971x^{11}+175.5908x^9-141.062x^7+66.287x^5-14.859*x^3+x)^2 dx \approx 0.0133732$

$M_1(T_{15}(89.154,1,0,1,x),1,89.154) = \int_{-1}^{1}(-0.00003658x^{25}+0.003521x^{23})^2 dx$
$= 5.17089e-7$

$(\frac{M_1(T_{15}(89.154,1,0,1,x),1,89.154)}{M_0(T_{15}(199.791,1,0,1,x))})^{\frac{1}{2}} = (\frac{5.17089e-7}{0.0133732})^{\frac{1}{2}} = 0.006218$

$(-\frac{d^2}{dx^2}+x^2-158.245)T_{23}(158.245,1,0,1,x) = -0.00004201x^{41}+0.007022x^{39}$

$M_0(T_{15}(158.245,1,0,1,x)) = \int_{-1}^{1}(-0.00004201x^{39}+0.0003744x^{37}-0.003014x^{35}+0.02178x^{33}-0.14029x^{31}+0.7992x^{29}-3.9917x^{27}+17.3015x^{25}-64.3216x^{23}+202.3247x^{21}-529.8335x^{19}+1132.8757x^{17}-1931.1526x^{15}+2546.9505x^{13}-2499.867x^{11}+1732.746x^9-787.064x^7+208.729x^5-26.374x^3+x)^2 dx \approx 0.11$

$M_1(T_{15}(158.245,1,0,1,x),1,158.245) = \int_{-1}^{1}(-0.00003658x^{25}+0.003521x^{23})^2 dx$
$= 5.17089e-7$

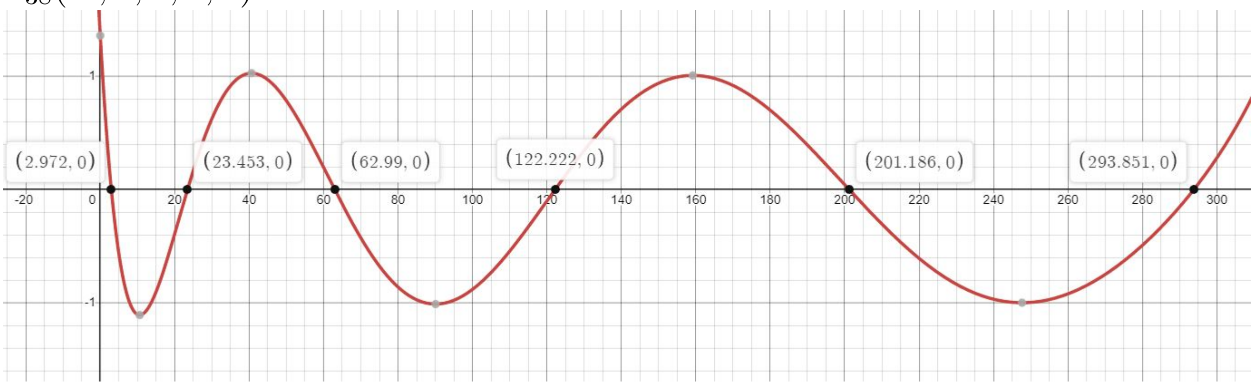
$(\frac{M_1(T_{15}(158.245,1,0,1,x),1,158.245)}{M_0(T_{15}(199.791,1,0,1,x))})^{\frac{1}{2}} = (\frac{5.17089e-7}{0.11})^{\frac{1}{2}} = 0.002168$

All these functions are very accurate.

4.9 Repeat all this process for other values of k

k=2:

$T_{38}(E, 2, 1, 0, 1):$



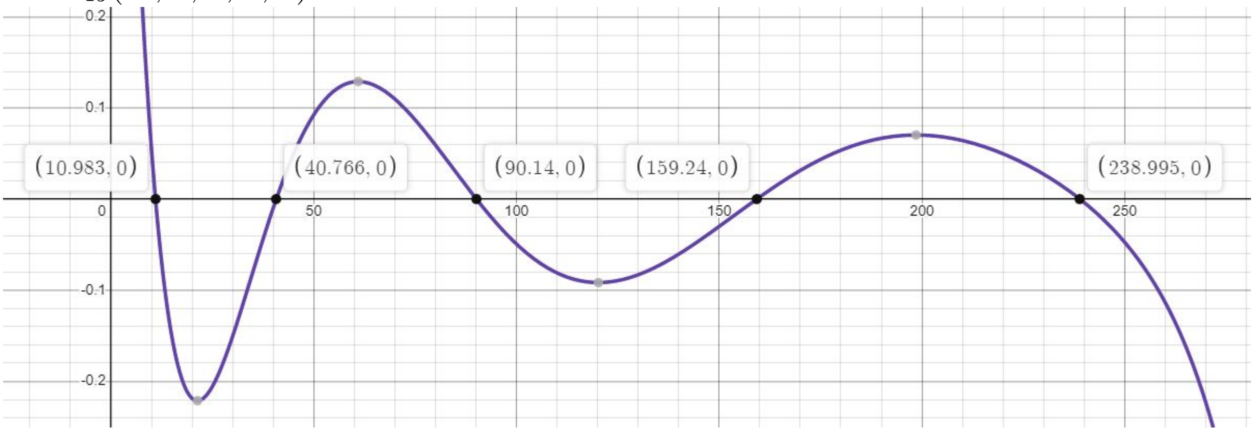
$T_{14}(2.972, 2, 1, 0, x) = 1 - 1.486x^2 + 0.7014x^4 - 0.2676x^6 + 0.0643x^8 - 0.01402x^{10} + 0.002264x^{12} - 0.000345x^{14} + 0.00004201x^{16} - 0.000004918x^{18}$

$T_{24}(23.453, 2, 1, 0, x) = 1 - 11.7265x^2 + 23.2518x^4 - 19.741x^6 + 9.928x^8 - 3.4646x^{10} + 0.9164x^{12} - 0.1942x^{14} + 0.03426x^{16} - 0.005165x^{18} + 0.0006793x^{20} - 0.0000792x^{22} + 0.000008288x^{24}$

$T_{32}(62.99, 2, 1, 0, x) = 1 - 31.495x^2 + 165.6558x^4 - 352.021x^6 + 407.79x^8 - 301.0557x^{10} + 156.02x^{12} - 60.615x^{14} + 18.509x^{16} - 4.602x^{18} + 0.9578x^{20} - 0.1704x^{22} + 0.02639x^{24} - 0.003606x^{26} + 0.0004401x^{28} - 4.844E - 05x^{30} + 4.8505E - 06x^{32}$

$T_{38}(122.222, 2, 1, 0, x) = 1 - 61.111x^2 + 622.7591x^4 - 2545.31003456257x^6 + 5599.7128x^8 + -7717.6594x^{10} + 7315.6562x^{12} - 5082.4438x^{14} + 2710.2044x^{16} - 1148.9424x^{18} + 398.0707x^{20} - 115.2571x^{22} + 28.40441x^{24} - 6.0503x^{26} + 1.1284x^{28} - 0.1863x^{30} + 0.02751x^{32} - 0.003661x^{34} + 0.0004425x^{36} - 4.8877E - 05x^{38}$

$T_{45}(E, 2, 0, 1, 1)$:



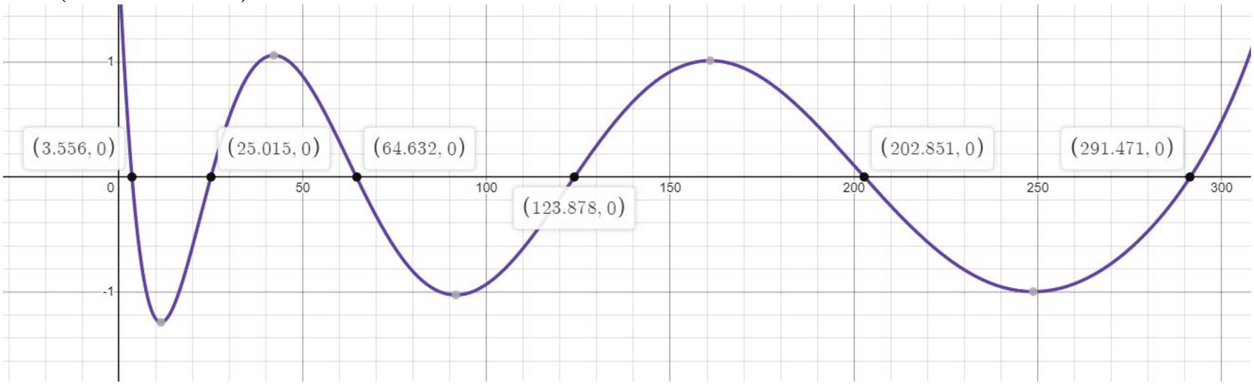
$T_{22}(10.983, 2, 0, 1, x) = x - 1.8305x^3 + 1.205x^5 - 0.4895x^7 + 0.1416x^9 - 0.03194x^{11} + 0.00588x^{13} - 0.0009159x^{15} + 0.0001235x^{17} - 1.4677E - 05x^{19} + 1.5596E - 06x^{21} - 1.499E - 07x^{23}$

$T_{29}(40.766, 2, 0, 1, x) = x - 6.7943x^3 + 14.0489x^5 - 14.2832x^7 + 8.8676x^9 - 3.8057x^{11} + 1.2219x^{13} - 0.3097x^{15} + 0.06438x^{17} - 0.01129x^{19} + 0.00171x^{21} - 0.00022704x^{23} + 2.6823E - 05x^{25} - 2.8513E - 06x^{27} + 2.7528E - 07x^{29}$

$T_{37}(90.14, 2, 0, 1, x) = x - 15.0233x^3 + 67.9102x^5 - 147.1789x^7 + 188.0326x^9 - 159.4362x^{11} + 96.9468x^{13} - 44.6502x^{15} + 16.2226x^{17} - 4.7978x^{19} + 1.1842x^{21} - 0.2489x^{23} + 0.04529x^{25} - 0.007233x^{27} + 0.001026x^{29} - 0.0001306x^{31} + 1.5031E - 05x^{33} - 1.5774E - 06x^{35} + 1.5189E - 07x^{37}$

k=3:

$T_{44}(E, 3, 1, 0, 1)$:

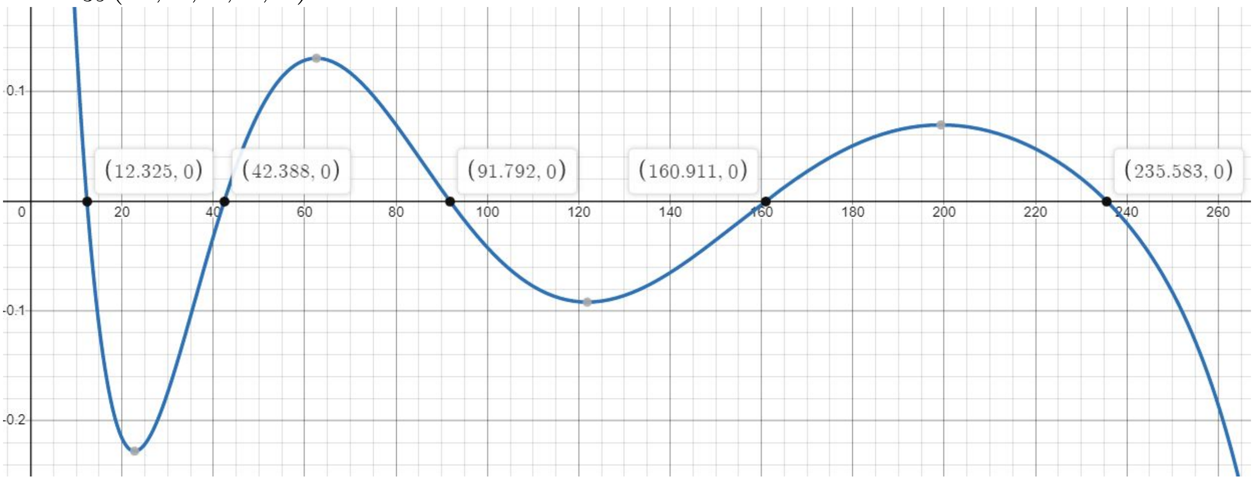


$T_{22}(3.556, 3, 1, 0, x) = 1 - 1.778x^2 + 1.2769x^4 - 0.6848x^6 + 0.2487x^8 - 0.0783x^{10} + 0.01907x^{12} - 0.004245x^{14} + 0.0007779x^{16} - 0.0001339x^{18} + 0.00001968x^{20} - 2.7595E - 06x^{22}$

$T_{28}(25.015, 3, 1, 0, x) = 1 - 12.508x^2 + 26.825x^4 - 26.121x^6 + 15.98x^8 - 7.0537x^{10} + 2.4263x^{12} - 0.6823x^{14} + 0.1621x^{16} - 0.03332x^{18} + 0.006033x^{20} - 0.0009758x^{22} + 0.0001426x^{24} - 0.000018998x^{26} + 0.0000023261x^{28}$

$T_{34}(64.632, 3, 1, 0, x) = 1 - 32.316x^2 + 174.804x^4 - 386.29x^6 + 473.93x^8 - 378.974x^{10} + 217.87x^{12} - 96.11x^{14} + 34.053x^{16} - 10.019x^{18} + 2.5107x^{20} - 0.5464x^{22} + 0.1049x^{24} - 0.017998x^{26} + 0.002788x^{28} - 0.0003933x^{30} + 5.091E - 05x^{32} - 6.087E - 06x^{34}$

$T_{39}(E,3,0,1,1):$



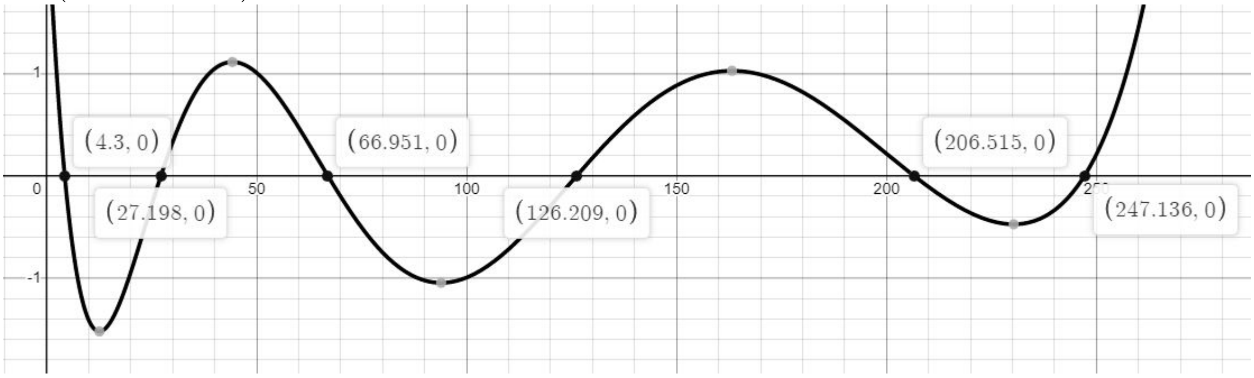
$T_{25}(12.325,3,0,1,1) = x - 2.0542x^3 + 1.716x^5 - 0.9437x^7 + 0.37603x^9 - 0.1193x^{11} + 0.03112x^{13} - 0.006941x^{15} + 0.001344x^{17} - 0.0002311x^{19} + 3.559E - 05x^{21} - 4.976E - 06x^{23} + 6.361E - 07x^{25}$

$T_{31}(42.388,3,0,1,1) = x - 7.0647x^3 + 15.4229x^5 - 17.07919x^7 + 11.9828x^9 - 6.0149x^{11} + 2.3257x^{13} - 0.7272x^{15} + 0.1903x^{17} - 0.04272x^{19} + 0.008389x^{21} - 0.001463x^{23} + 0.0002292x^{25} - 0.000032589x^{27} + 4.2412E - 06x^{29} - 5.08679E - 07x^{31}$

$T_{39}(91.792,3,0,1,1) = x - 15.2987x^3 + 70.6648x^5 - 157.7178x^7 + 209.9058x^9 - 188.0648x^{11} + 122.76922x^{13} - 61.7229x^{15} + 24.8919x^{17} - 8.3052x^{19} + 2.3485x^{21} - 0.5738x^{23} + 0.123x^{25} - 0.02344x^{27} + 0.004013x^{29} - 0.0006229x^{31} + 8.8351E - 05x^{33} - 1.1526E - 05x^{35} + 1.3913E - 06x^{37} - 1.5617E - 07x^{39}$

k=4:

$T_{40}(E, 4, 1, 0, 1)$:

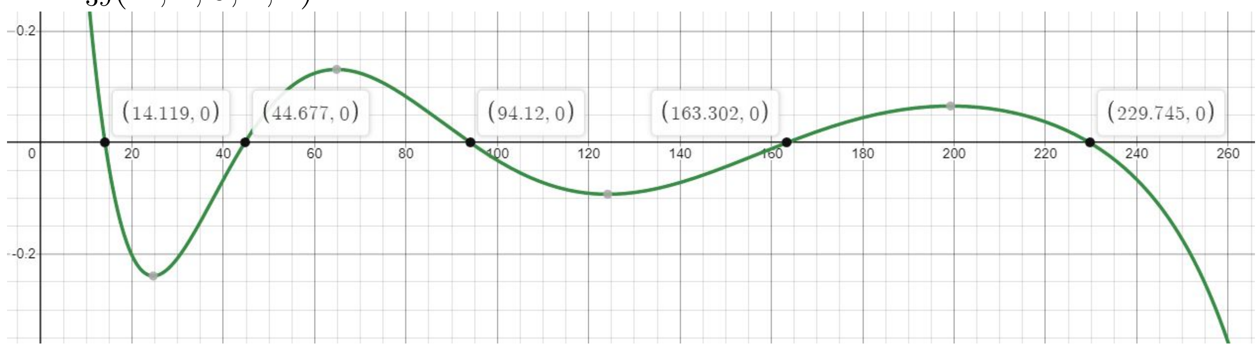


$T_{28}(4.3, 4, 1, 0, x) = 1 - 2.15x^2 + 2.104x^4 - 1.448x^6 + 0.7123x^8 - 0.2915x^{10} + 0.09583x^{12} - 0.02789x^{14} + 0.006888x^{16} - 0.001555x^{18} + 0.0003076x^{20} - 5.672E - 05x^{22} + 9.359E - 06x^{24} - 0.000001458x^{26} + 2.064E - 07x^{28}$

$T_{32}(27.198, 4, 1, 0, x) = 1 - 13.599x^2 + 32.1555x^4 + -36.4049x^6 + 26.8684x^8 - 14.5916x^{10} + 6.2633x^{12} + -2.2188x^{14} + 0.669x^{16} - 0.1755x^{18} + 0.04073x^{20} - 0.008475x^{22} + 0.001598x^{24} - 0.0002755x^{26} + 4.3732E - 05x^{28} - 6.4334E - 06x^{30} + 8.8175E - 07x^{32}$

$T_{38}(66.951, 4, 1, 0, x) = 1 - 33.4755x^2 + 188.1015x^4 - 437.6398x^6 + 576.9651x^8 - 507.0069x^{10} + 327.0914x^{12} - 164.8967x^{14} + 67.8061x^{16} - 23.4576x^{18} + 6.9879x^{20} - 1.825x^{22} + 0.4239x^{24} - 0.08859x^{26} + 0.01682x^{28} - 0.002923x^{30} + 0.0004685x^{32} - 6.9645E - 05x^{34} + 9.6503E - 06x^{36} - 1.2521E - 06x^{38}$

$T_{39}(E, 4, 0, 1, 1)$:



$T_{29}(14.119, 4, 0, 1, 1) = x - 2.353x^3 + 2.461x^5 - 1.724x^7 + 0.88497x^9 - 0.3643x^{11} + 0.1237x^{13} - 0.03608x^{15} + 0.009152x^{17} - 0.002066x^{19} + 0.0004181x^{21} - 7.698E - 05x^{23} + 1.296E - 05x^{25} - 2.015E - 06x^{27} + 2.904E - 07x^{29}$

$T_{35}(44.677, 4, 0, 1, 1) = x - 7.4462x^3 + 17.4336x^5 - 21.3814x^7 + 17.1416x^9 + -10.0722x^{11} + 4.6427x^{13} - 1.7551x^{15} + 0.5614x^{17} - 0.1554x^{19} + 0.0379x^{21} - 0.008264x^{23} + 0.001627x^{25} - 0.0002919x^{27} + 4.8109E - 05x^{29} - 7.3325E - 06x^{31} + 1.03915E - 06x^{33} - 1.376E - 07x^{35}$

$T_{39}(94.12, 4, 0, 1, 1) = x - 15.6867x^3 + 74.6215x^5 - 173.199x^7 + 242.9921x^9 - 233.1055x^{11} + 165.5626x^{13} - 91.964x^{15} + 41.5612x^{17} - 15.7403x^{19} + 5.1106x^{21} - 1.4483x^{23} + 0.3635x^{25} - 0.08174x^{27} + 0.01664x^{29} - 0.00309x^{31} + 0.0005275x^{33} - 8.3268E - 05x^{35} + 1.222E - 05x^{37} - 1.6751E - 06x^{39}$

4.10 Measuring the accuracies

$(-\frac{d^2}{dx^2}+x^2-2.972)T_{18}(2.972,1,1,0,x) = -0.000004918x^{20}+0.003536x^{18}$

$M_0(T_{15}(2.972,1,1,0,x)) = \int_{-1}^{1}(1-1.486x^2+0.7014x^4-0.2676x^6+0.0643x^8-0.01402x^{10}+0.002264x^{12}-0.000345x^{14}+0.00004201x^{16}-0.000004918x^{18})^2dx \approx 0.946456$

$M_1(T_{15}(2.972,1,1,0,x),1,2.972) = \int_{-1}^{1}(-0.000004918x^{20}+0.003536x^{18})^2dx = 6.74071e-7$

$(\frac{M_1(T_{15}(2.972,1,1,0,x),1,2.972)}{M_0(T_{15}(199.791,1,1,0,x))})^{\frac{1}{2}} = (\frac{6.74071e-7}{0.94645})^{\frac{1}{2}} = 0.0008439$

$(-\frac{d^2}{dx^2}+x^2-23.453)T_{24}(23.453,2,1,0,x) = 0.000008288x^{26}-0.0002736x^{24}$

$M_0(T_{20}(23.453,2,1,0,x)) = \int_{-1}^{1}(1-11.7265x^2+23.2518x^4-19.741x^6+9.928x^8-3.4646x^{10}+0.9164x^{12}-0.1942x^{14}+0.03426x^{16}-0.005165x^{18}+0.0006793x^{20}-0.0000792x^{22}+0.000008288x^{24})^2dx \approx 1.02503$

$M_1(T_{20}(23.453,2,1,0,x),1,23.453) = \int_{-1}^{1}(0.000008288x^{26}-0.0002736x^{24})^2dx = 2.88013e-9$

$(\frac{M_1(T_{20}(23.453,2,1,0,x),1,23.453)}{M_0(T_{15}(199.791,1,1,0,x))})^{\frac{1}{2}} = (\frac{2.88013e-9}{1.02503})^{\frac{1}{2}} = 0.0000530076$

$(-\frac{d^2}{dx^2}+x^2-62.99)T_{32}(62.99,2,1,0,x) = 4.8505E-06x^{34}-0.00035397x^{32}$

$M_0(T_{20}(62.99,2,1,0,x)) = \int_{-1}^{1}(1-31.495x^2+165.6558x^4-352.021x^6+407.79x^8-301.0557x^{10}+156.02x^{12}-60.615x^{14}+18.509x^{16}-4.602x^{18}+0.9578x^{20}-0.1704x^{22}+0.02639x^{24}-0.003606x^{26}+0.0004401x^{28}-4.844E-05x^{30}+4.8505E-06x^{32})^2dx \approx 1.00979$

$M_1(T_{20}(62.99,2,1,0,x),1,62.99) = \int_{-1}^{1}(4.8505E-06x^{34}-0.00035397x^{32})^2dx = 3.7534e-9$

$(\frac{M_1(T_{20}(62.99,2,1,0,x),1,62.99)}{M_0(T_{15}(199.791,1,1,0,x))})^{\frac{1}{2}} = (\frac{3.7534e-9}{1.00979})^{\frac{1}{2}} = 0.0000609673$

$(-\frac{d^2}{dx^2}+x^2-122.222)T_{38}(122.222,2,1,0,x) = -4.8877E{-}05x^{40}-0.00553x^{38}$

$M_0(T_{20}(122.222,2,1,0,x)) = \int_{-1}^{1}(1-61.111x^2+622.7591x^4-2545.31003456257x^6+5599.7128x^8+-7717.6594x^{10}+7315.6562x^{12}-5082.4438x^{14}+2710.2044x^{16}-1148.9424x^{18}+398.0707x^{20}-115.2571x^{22}+28.40441x^{24}-6.0503x^{26}+1.1284x^{28}-0.1863x^{30}+0.02751x^{32}-0.003661x^{34}+0.0004425x^{36}-4.8877E{-}05x^{38})^2dx \approx 0.918346$

$M_1(T_{20}(122.222,2,1,0,x),1,122.222) = \int_{-1}^{1}(-4.8877E{-}05x^{40}-0.00553x^{38})^2dx = 8.08054e-7$

$(\frac{M_1(T_{20}(122.222,2,1,0,x),1,122.222)}{M_0(T_{15}(199.791,1,1,0,x))})^{\frac{1}{2}} = (\frac{8.08054e-7}{0.918346})^{\frac{1}{2}} = 0.000938031$

$(-\frac{d^2}{dx^2}+x^2-10.983)T_{22}(10.983,2,1,0,x) = -1.499E{-}07x^{25}+0.000003206x^{27}$

$M_0(T_{22}(10.983,2,1,0,x)) = \int_{-1}^{1}(x-1.8305x^3+1.205x^5-0.4895x^7+0.1416x^9-0.03194x^{11}+0.00588x^{13}-0.0009159x^{15}+0.0001235x^{17}-1.4677E-05x^{19}+1.5596E-06x^{21}-1.499E-07x^{23})^2dx \approx 0.092941$

$M_1(T_{22}(10.983,2,1,0,x),1,10.983) = \int_{-1}^{1}(-1.499E{-}07x^{25}+0.000003206x^{27})^2dx = 3.38372e-13$

$(\frac{M_1(T_{22}(10.983,2,1,0,x),1,10.983)}{M_0(T_{15}(199.791,1,1,0,x))})^{\frac{1}{2}} = (\frac{3.38372e-13}{0.092941})^{\frac{1}{2}} = 1.90807e-6$

$(-\frac{d^2}{dx^2}+x^2-40.766)T_{29}(40.766,2,0,1,x) = 2.7528E{-}07x^{25}-0.00001407x^{27}$

$M_0(T_{29}(40.766,2,0,1,x)) = \int_{-1}^{1}(x-6.7943x^3+14.0489x^5-14.2832x^7+8.8676x^9-3.8057x^{11}+1.2219x^{13}-0.3097x^{15}+0.06438x^{17}-0.01129x^{19}+0.00171x^{21}-0.00022704x^{23}+2.6823E{-}05x^{25}-2.8513E{-}06x^{27}+2.7528E{-}07x^{29})^2dx \approx 0.0248758$

$M_1(T_{29}(40.766,2,0,1,x),1,40.766) = \int_{-1}^{1}(2.7528E{-}07x^{25}-0.00001407x^{27})^2dx = 6.90938e-12$

$(\frac{M_1(T_{29}(40.766,2,0,1,x),1,40.766)}{M_0(T_{29}(40.766,2,0,1,x))})^{\frac{1}{2}} = (\frac{6.90938e-12}{0.0248758})^{\frac{1}{2}} = 0.00001667$

$$(-\frac{d^2}{dx^2}+x^2-90.14)T_{37}(90.14,2,0,1,x) = 1.5189E-07x^{39}-0.00001527x^{27}$$

$$M_0(T_{37}(90.14,2,0,1,x)) = \int_{-1}^{1}(x-15.0233x^3+67.9102x^5-147.1789x^7+188.0326x^9-159.4362x^{11}+96.9468x^{13}-44.6502x^{15}+16.2226x^{17}-4.7978x^{19}+1.1842x^{21}-0.2489x^{23}+0.04529x^{25}-0.007233x^{27}+0.001026x^{29}-0.0001306x^{31}+1.5031E-05x^{33}-1.5774E-06x^{35}+1.5189E-07x^{37})^2dx \approx 0.0705178$$

$$M_1(T_{37}(90.14,2,0,1,x),1,90.14) = \int_{-1}^{1}(1.5189E-07x^{39}-0.00001527x^{27})^2dx = 8.34113e-12$$

$$(\frac{M_1(T_{37}(90.14,2,0,1,x),1,90.14)}{M_0(T_{37}(90.14,2,0,1,x))})^{\frac{1}{2}} = (\frac{8.34113e-12}{0.0705178})^{\frac{1}{2}} = 0.00001088$$

$$(-\frac{d^2}{dx^2}+x^2-3.556)T_{22}(3.556,3,1,0,x) = -2.7595E-06x^{24}+0.00002949x^{22}$$

$$M_0(T_{22}(3.556,3,1,0,x)) = \int_{-1}^{1}(1-1.778x^2+1.2769x^4-0.6848x^6+0.2487x^8-0.0783x^{10}+0.01907x^{12}-0.004245x^{14}+0.0007779x^{16}-0.0001339x^{18}+0.00001968x^{20}-2.7595E-06x^{22})^2dx = 0.890887$$

$$M_1(T_{22}(3.556,3,1,0,x),1,3.556) = \int_{-1}^{1}(-2.7595E-06x^{24}+0.00002949x^{22})^2dx = 3.20366e-11$$

$$(\frac{M_1(T_{22}(3.556,3,1,0,x),1,3.556)}{M_0(T_{22}(3.556,3,1,0,x))})^{\frac{1}{2}} = (\frac{3.20366e-11}{0.890887})^{\frac{1}{2}} = 5.9967e-6$$

$$(-\frac{d^2}{dx^2}+x^2-25.015)T_{28}(25.015,3,1,0,x) = 0.0000023261x^{30}-0.000077185x^{28}$$

$$M_0(T_{28}(25.015,3,1,0,x)) = \int_{-1}^{1}(1-12.508x^2+26.825x^4-26.121x^6+15.98x^8-7.0537x^{10}+2.4263x^{12}-0.6823x^{14}+0.1621x^{16}-0.03332x^{18}+0.006033x^{20}-0.0009758x^{22}+0.0001426x^{24}-0.000018998x^{26}+0.0000023261x^{28})^2dx = 1.05298$$

$$M_1(T_{28}(25.015,3,1,0,x),1,25.015) = \int_{-1}^{1}(0.0000023261x^{30}-0.000077185x^{28})^2dx = 1.97041e-10$$

$$(\frac{M_1(T_{28}(25.015,3,1,0,x),1,25.015)}{M_0(T_{28}(25.015,3,1,0,x))})^{\frac{1}{2}} = (\frac{1.97041e-10}{1.05298})^{\frac{1}{2}} = 0.0000136794$$

$(-\frac{d^2}{dx^2}+x^2-64.632)T_{34}(64.632,3,1,0,x) = -6.087E-06x^{36}+0.0004443x^{28}$

$M_0(T_{34}(64.632,3,1,0,x)) = \int_{-1}^{1}(1-32.316x^2+174.804x^4-386.29x^6+473.93x^8-378.974x^{10}+217.87x^{12}-96.11x^{14}+34.053x^{16}-10.019x^{18}+2.5107x^{20}-0.5464x^{22}+0.1049x^{24}-0.017998x^{26}+0.002788x^{28}-0.0003933x^{30}+5.091E-05x^{32}-6.087E-06x^{34})^2dx = 1.1795$

$M_1(T_{34}(64.632,3,1,0,x),1,64.632) = \int_{-1}^{1}(-6.087E-06x^{36}+0.0004443x^{28})^2dx = 6.76099e-9$

$(\frac{M_1(T_{34}(64.632,3,1,0,x),1,64.632)}{M_0(T_{34}(64.632,3,1,0,x))})^{\frac{1}{2}} = (\frac{6.76099e-9}{1.1795})^{\frac{1}{2}} = 0.0000757105$

$(-\frac{d^2}{dx^2}+x^2-12.325)T_{25}(12.325,3,0,1,1) = 6.361E-07x^{27}-0.00001282x^{25}$

$M_0(T_{25}(12.325,3,0,1,1)) = \int_{-1}^{1}(x-2.0542x^3+1.716x^5-0.9437x^7+0.37603x^9-0.1193x^{11}+0.03112x^{13}-0.006941x^{15}+0.001344x^{17}-0.0002311x^{19}+3.559E-05x^{21}-4.976E-06x^{23}+6.361E-07x^{25})^2dx = 0.0839517$

$M_1(T_{25}(12.325,3,0,1,1),1,12.325) = \int_{-1}^{1}(6.361E-07x^{27}-0.00001282x^{25})^2dx = 5.84445e-12$

$(\frac{M_1(T_{25}(12.325,3,0,1,1),1,12.325)}{M_0(T_{25}(12.325,3,0,1,1))})^{\frac{1}{2}} = (\frac{5.84445e-12}{0.0839517})^{\frac{1}{2}} = 8.34367e-6$

$(-\frac{d^2}{dx^2}+x^2-42.388)T_{31}(42.388,3,0,1,1) = -5.08679E-07x^{33}+0.000025803x^{31}$

$M_0(T_{31}(42.388,3,0,1,1)) = \int_{-1}^{1}(x-7.0647x^3+15.4229x^5-17.07919x^7+11.9828x^9-6.0149x^{11}+2.3257x^{13}-0.7272x^{15}+0.1903x^{17}-0.04272x^{19}+0.008389x^{21}-0.001463x^{23}+0.0002292x^{25}-0.000032589x^{27}+4.2412E-06x^{29}-5.08679E-07x^{31})^2dx = 0.0243158$

$M_1(T_{31}(42.388,3,0,1,1),1,42.388) = \int_{-1}^{1}(-5.08679E-07x^{33}+0.000025803x^{31})^2dx = 2.03363e-11$

$(\frac{M_1(T_{31}(42.388,3,0,1,1),1,42.388)}{M_0(T_{31}(42.388,3,0,1,1))})^{\frac{1}{2}} = (\frac{2.03363e-11}{0.0243158})^{\frac{1}{2}} = 0.0000289196$

$(-\frac{d^2}{dx^2}+x^2-91.792)T_{39}(91.792,3,0,1,1) = -1.5617E-07x^{41}+0.000015726x^{39}$

$M_0(T_{39}(91.792,3,0,1,1)) = \int_{-1}^{1}(x - 15.2987x^3 + 70.6648x^5 - 157.7178x^7 + 209.9058x^9-188.0648x^{11}+122.76922x^{13}-61.7229x^{15}+24.8919x^{17}-8.3052x^{19}+2.3485x^{21}-0.5738x^{23}+0.123x^{25}-0.02344x^{27}+0.004013x^{29}-0.0006229x^{31}+8.8351E-05x^{33}-1.1526E-05x^{35}+1.3913E-06x^{37}-1.5617E-07x^{39})^2dx \approx 0.0251766$

$M_1(T_{39}(91.792,3,0,1,1),1,91.792) = \int_{-1}^{1}(-1.5617E-07x^{41}+0.000015726x^{39})^2dx = 6.14025e-12$

$(\frac{M_1(T_{39}(91.792,3,0,1,1),1,91.792)}{M_0(T_{39}(91.792,3,0,1,1))})^{\frac{1}{2}} = (\frac{6.14025e-12}{0.0251766})^{\frac{1}{2}} = 0.0000312338$

$(-\frac{d^2}{dx^2} + x^2 - 4.3)T_{28}(4.3,4,1,0,x) = 2.064E - 07x^{30} - 0.000002346x^{28}$

$M_0(T_{28}(4.3,4,1,0,x)) = \int_{-1}^{1}(1 - 2.15x^2 + 2.104x^4 - 1.448x^6 + 0.7123x^8 - 0.2915x^{10}+0.09583x^{12}-0.02789x^{14}+0.006888x^{16}-0.001555x^{18}+0.0003076x^{20}-5.672E-05x^{22}+9.359E-06x^{24}-0.000001458x^{26}+2.064E-07x^{28})^2dx = 0.828978$

$M_1(T_{28}(4.3,4,1,0,x),1,4.3) = \int_{-1}^{1}(2.064E - 07x^{30} - 0.000002346x^{28})^2dx = 1.61681e - 13$

$(\frac{M_1(T_{28}(4.3,4,1,0,x),1,4.3)}{M_0(T_{28}(4.3,4,1,0,x))})^{\frac{1}{2}} = (\frac{1.61681e-13}{0.828978})^{\frac{1}{2}} = 4.41629e - 7$

$(-\frac{d^2}{dx^2}+x^2-27.198)T_{32}(27.198,4,1,0,x) = 8.8175E-07x^{34}-0.000030437x^{32}$

$M_0(T_{32}(27.198,4,1,0,x)) = \int_{-1}^{1}(1 - 13.599x^2 + 32.1555x^4 + -36.4049x^6 + 26.8684x^8 - 14.5916x^{10} + 6.2633x^{12} + -2.2188x^{14} + 0.669x^{16} - 0.1755x^{18} + 0.04073x^{20} - 0.008475x^{22} + 0.001598x^{24} - 0.0002755x^{26} + 4.3732E-05x^{28} - 6.4334E - 06x^{30} + 8.8175E - 07x^{32})^2dx \approx 1.08521$

$M_1(T_{32}(27.198,4,1,0,x),1,27.198) = \int_{-1}^{1}(8.8175E-07x^{34}-0.000030437x^{32})^2dx = 2.69252e - 11$

$(\frac{M_1(T_{32}(27.198,4,1,0,x),1,27.198)}{M_0(T_{32}(27.198,4,1,0,x))})^{\frac{1}{2}} = (\frac{2.69252e-11}{1.08521})^{\frac{1}{2}} = 4.98107e - 6$

$(-\frac{d^2}{dx^2}+x^2-66.951)T_{38}(66.951,4,1,0,x) = -1.2521E-06x^{40}+0.00009348x^{38}$

$M_0(T_{38}(66.951,4,1,0,x)) = \int_{-1}^{1}(1-33.4755x^2+188.1015x^4-437.6398x^6+576.9651x^8-507.0069x^{10}+327.0914x^{12}-164.8967x^{14}+67.8061x^{16}-23.4576x^{18}+6.9879x^{20}-1.825x^{22}+0.4239x^{24}-0.08859x^{26}+0.01682x^{28}-0.002923x^{30}+0.0004685x^{32}-6.9645E-05x^{34}+9.6503E-06x^{36}-1.2521E-06x^{38})^2dx \approx 1.14882$

$M_1(T_{38}(66.951,4,1,0,x),1,66.951) = \int_{-1}^{1}(-1.2521E-06x^{40}+0.00009348x^{38})^2dx = 2.21087e-10$

$(\frac{M_1(T_{38}(66.951,4,1,0,x),1,66.951)}{M_0(T_{38}(66.951,4,1,0,x))})^{\frac{1}{2}} = (\frac{2.21087e-10}{1.14882})^{\frac{1}{2}} = 0.0000138725$

$(-\frac{d^2}{dx^2}+x^2-14.119)T_{29}(14.119,4,0,1,1) = 2.904E-07x^{31}-0.000006115x^{29}$

$M_0(T_{29}(14.119,4,0,1,1)) = \int_{-1}^{1}(x-2.353x^3+2.461x^5-1.724x^7+0.88497x^9-0.3643x^{11}+0.1237x^{13}-0.03608x^{15}+0.009152x^{17}-0.002066x^{19}+0.0004181x^{21}-7.698E-05x^{23}+1.296E-05x^{25}-2.015E-06x^{27}+2.904E-07x^{29})^2dx = 0.0734848$

$M_1(T_{29}(14.119,4,0,1,1),1,14.119) = \int_{-1}^{1}(2.904E-07x^{31}-0.000006115x^{29})^2dx = 1.1538e-12$

$(\frac{M_1(T_{29}(14.119,4,0,1,1),1,14.119)}{M_0(T_{29}(14.119,4,0,1,1))})^{\frac{1}{2}} = (\frac{1.1538e-12}{0.0734848})^{\frac{1}{2}} = 3.96247e-6$

$(-\frac{d^2}{dx^2}+x^2-44.677)T_{35}(44.677,4,0,1,1) = -1.376E-07x^{37}+0.000007187x^{35}$

$M_0(T_{35}(44.677,4,0,1,1)) = \int_{-1}^{1}(x-7.4462x^3+17.4336x^5-21.3814x^7+17.1416x^9+-10.0722x^{11}+4.6427x^{13}-1.7551x^{15}+0.5614x^{17}-0.1554x^{19}+0.0379x^{21}-0.008264x^{23}+0.001627x^{25}-0.0002919x^{27}+4.8109E-05x^{29}-7.3325E-06x^{31}+1.03915E-06x^{33}-1.376E-07x^{35})^2dx = 0.0296724$

$M_1(T_{35}(44.677,4,0,1,1),1,44.677) = \int_{-1}^{1}(-1.376E-07x^{37}+0.000007187x^{35})^2dx = 1.40133e-12$

$(\frac{M_1(T_{35}(44.677,4,0,1,1),1,44.677)}{M_0(T_{35}(44.677,4,0,1,1))})^{\frac{1}{2}} = (\frac{1.40133e-12}{0.0296724})^{\frac{1}{2}} = 6.87217e-6$

$(-\frac{d^2}{dx^2}+x^2-94.12)T_{39}(94.12,4,0,1,1) = -1.6751E-06x^{41}+0.00016988x^{39}$

$M_0(T_{39}(94.12,4,0,1,1)) = \int_{-1}^{1}(x - 15.6867x^3 + 74.6215x^5 - 173.199x^7 + 242.9921x^9 - 233.1055x^{11} + 165.5626x^{13} - 91.964x^{15} + 41.5612x^{17} - 15.7403x^{19} + 5.1106x^{21} - 1.4483x^{23} + 0.3635x^{25} - 0.08174x^{27} + 0.01664x^{29} - 0.00309x^{31} + 0.0005275x^{33} - 8.3268E-05x^{35} + 1.222E-05x^{37} - 1.6751E-06x^{39})^2 dx \approx 0.0344058$

$M_1(T_{39}(94.12,4,0,1,1),1,94.12) = \int_{-1}^{1}(-1.6751E-06x^{41}+0.00016988x^{39})^2 dx = 7.16628e-10$

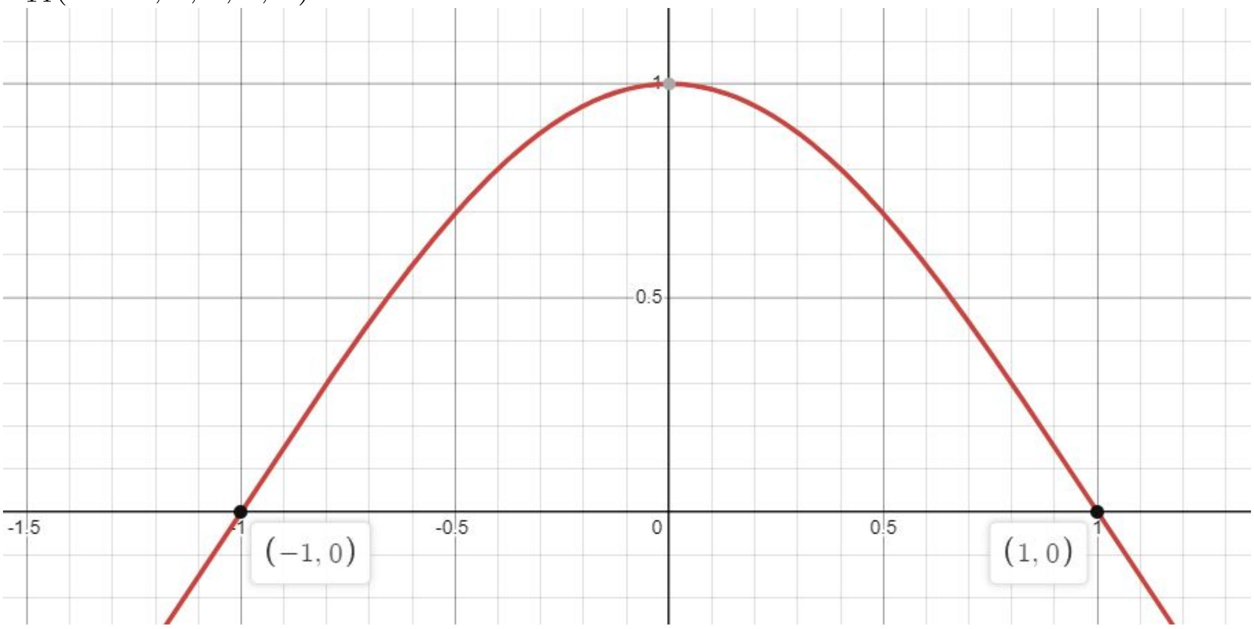
$(\frac{M_1(T_{39}(94.12,4,0,1,1),1,94.12)}{M_0(T_{39}(94.12,4,0,1,1))})^{\frac{1}{2}} = (\frac{7.16628e-10}{0.0344058})^{\frac{1}{2}} = 0.000144322$

We can see that all our functions are very accurate.

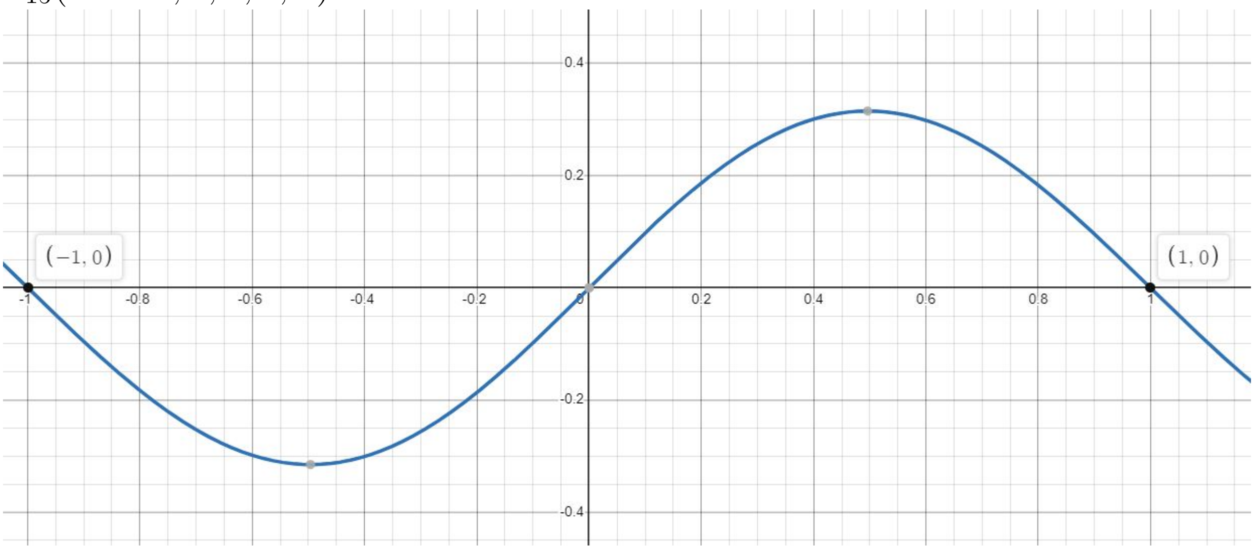
5 Graphs of all accurate nonzero eigenfunctions we found

5.1 k=1

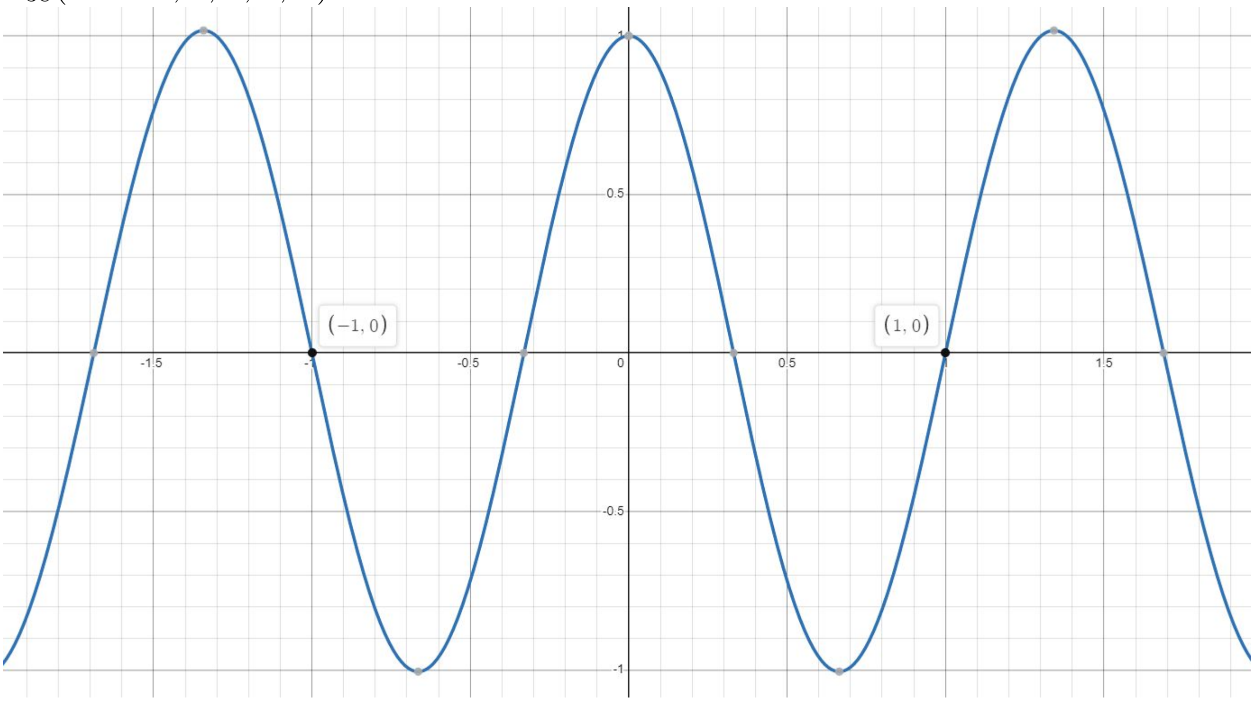
$T_{14}(2.597, 1, 1, 0, x)$:



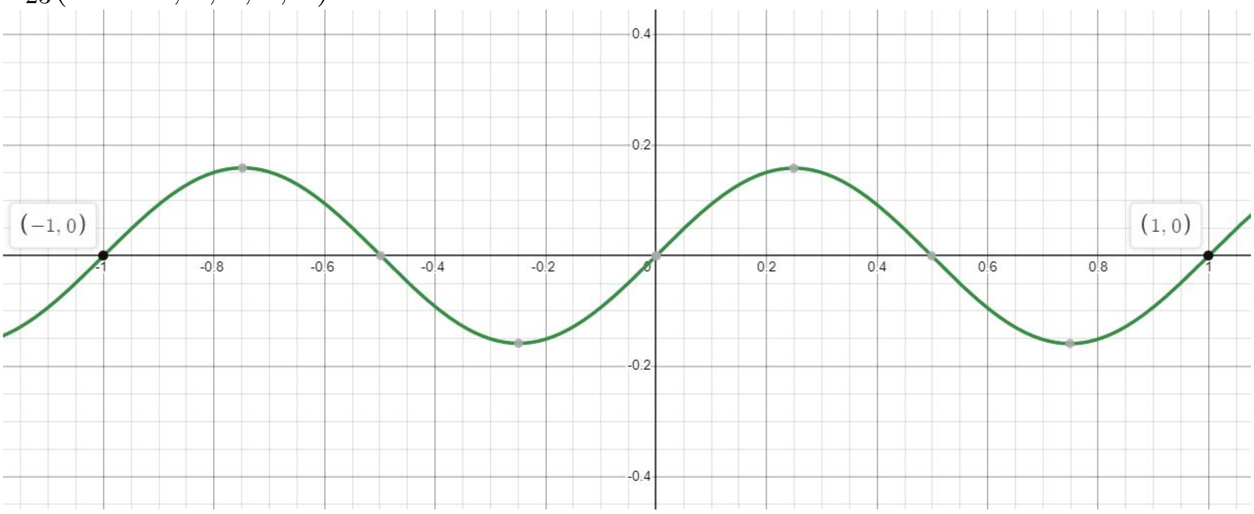
$T_{15}(10.151, 1, 0, 1, x)$:



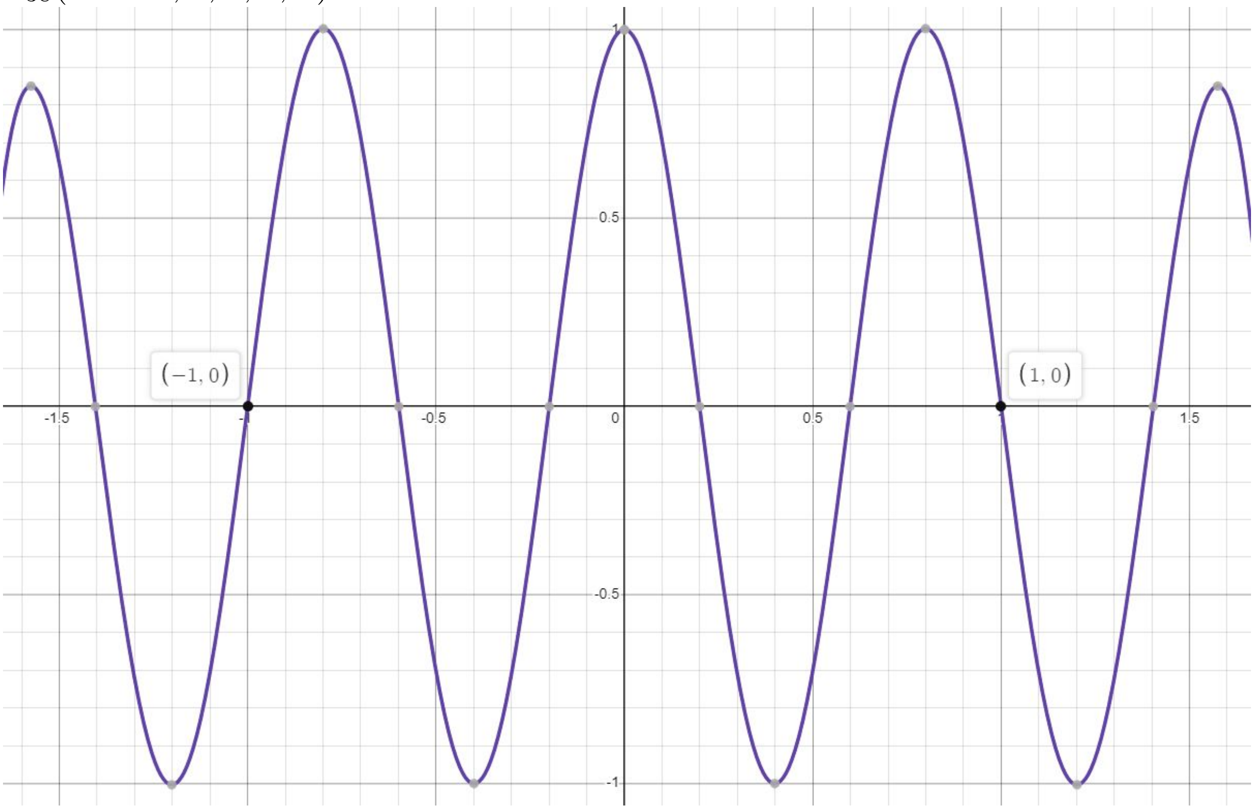
$T_{38}(22.518, 1, 1, 0, x)$:



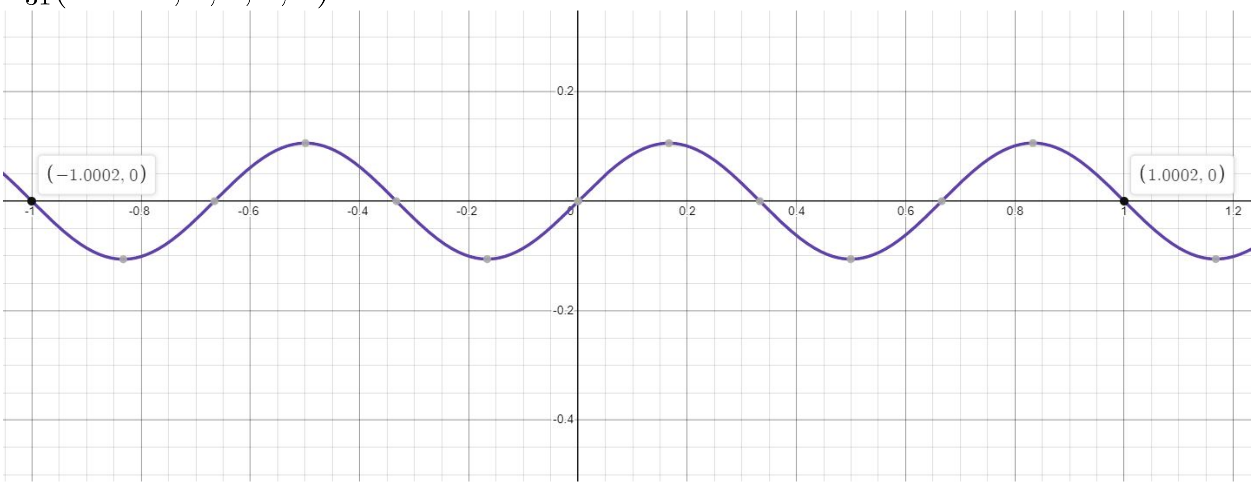
$T_{23}(39.799, 1, 0, 1, x)$:



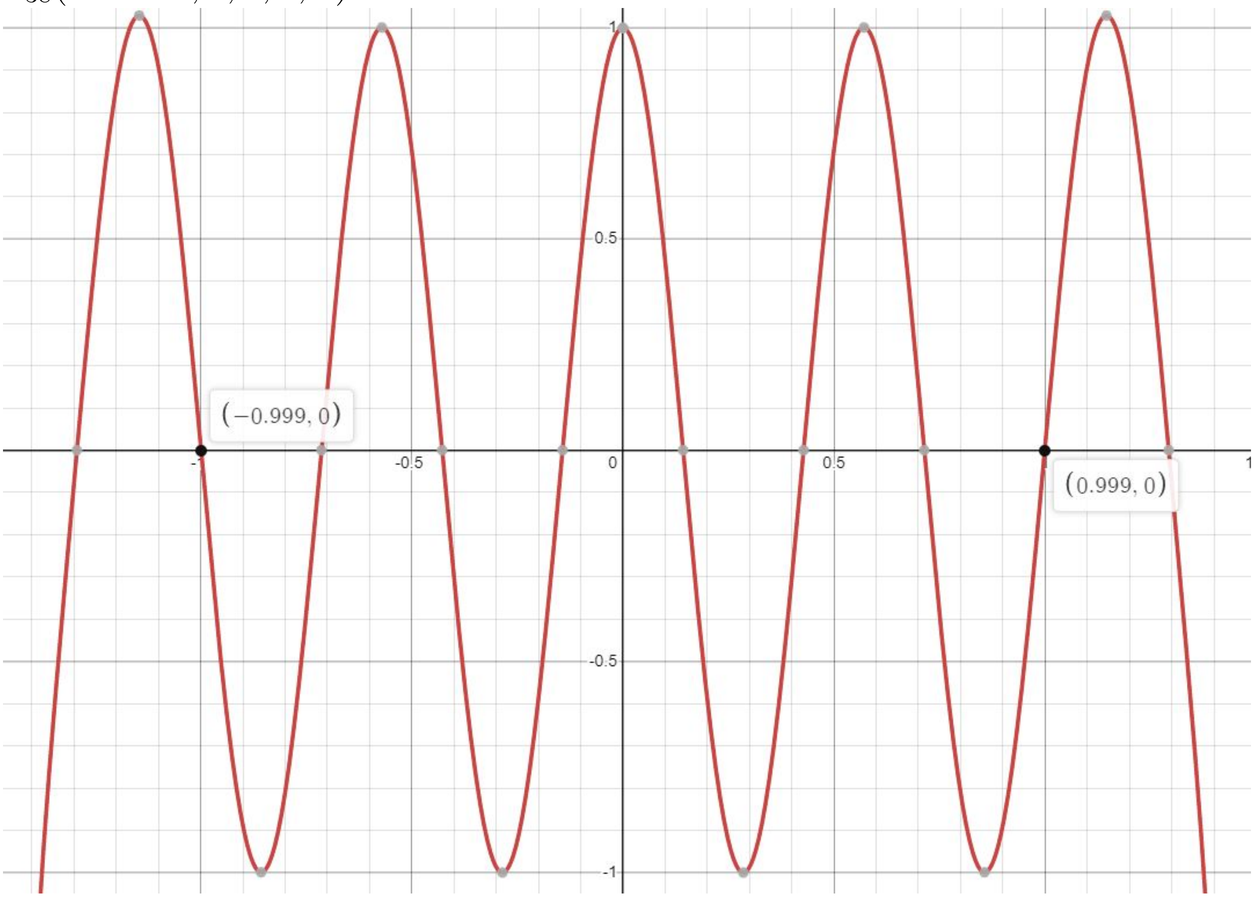
$T_{38}(62.011, 1, 1, 0, x)$:



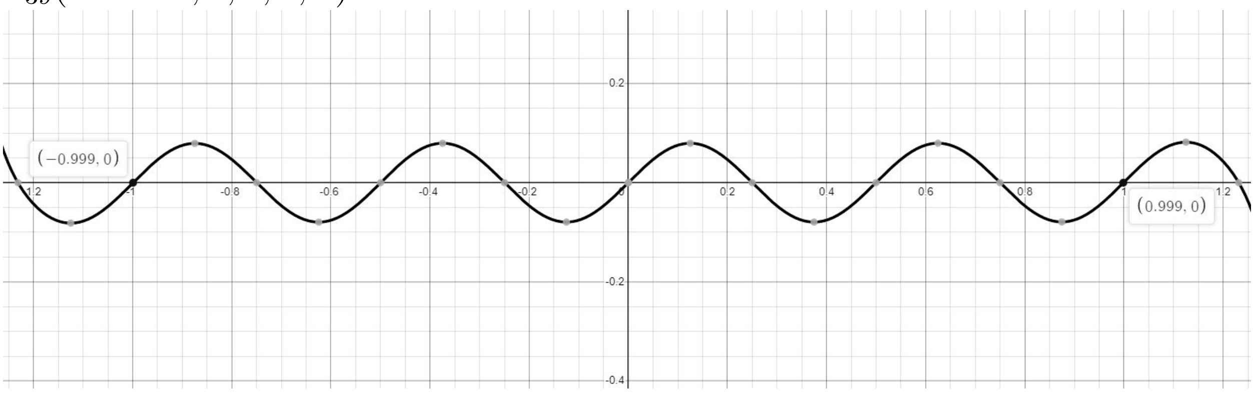
$T_{31}(89.154, 1, 0, 1, x)$:



$T_{38}(121.232, 1, 1, 0, x)$:

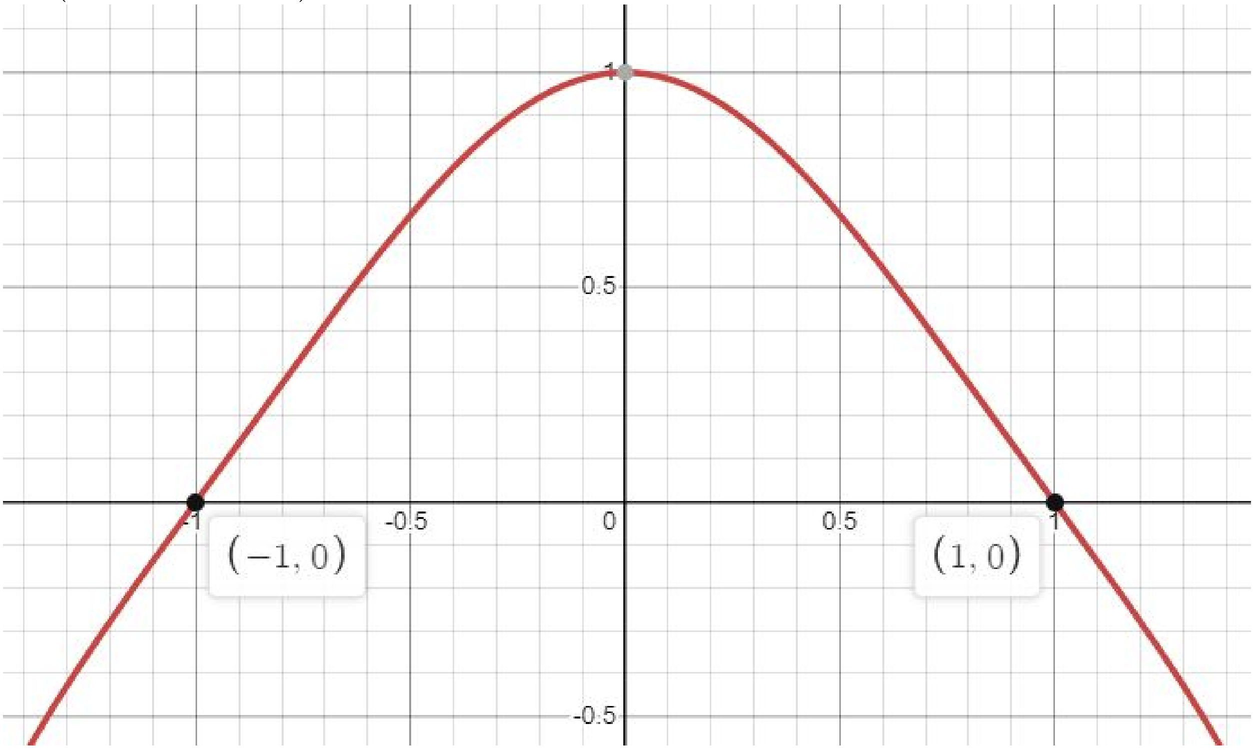


$T_{39}(158.245, 1, 0, 1, x)$:

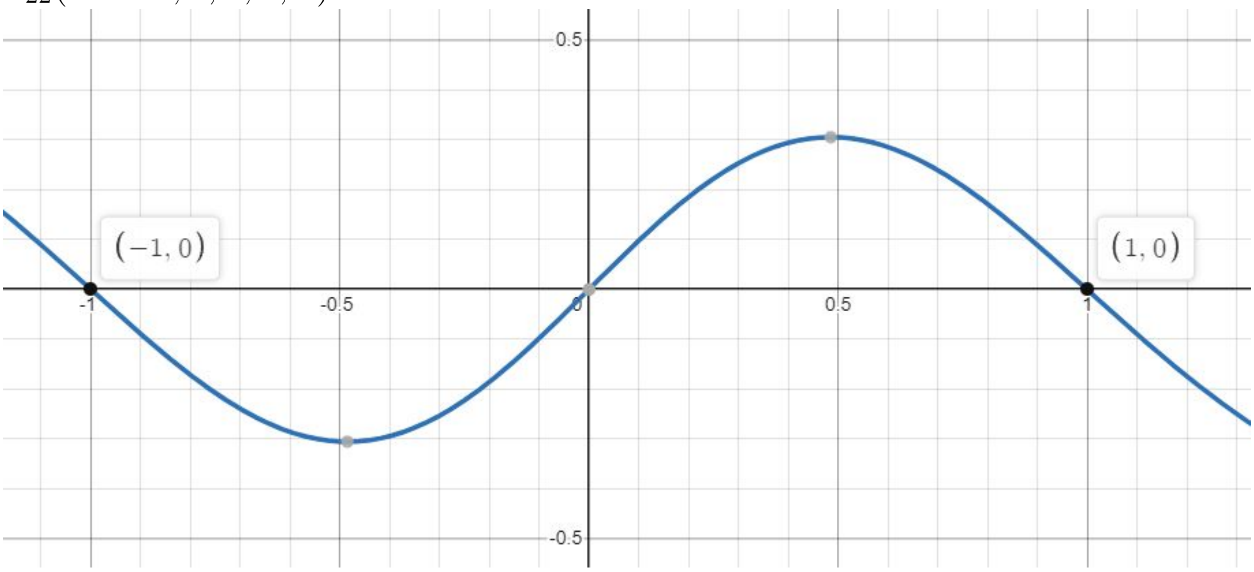


5.2 k=2

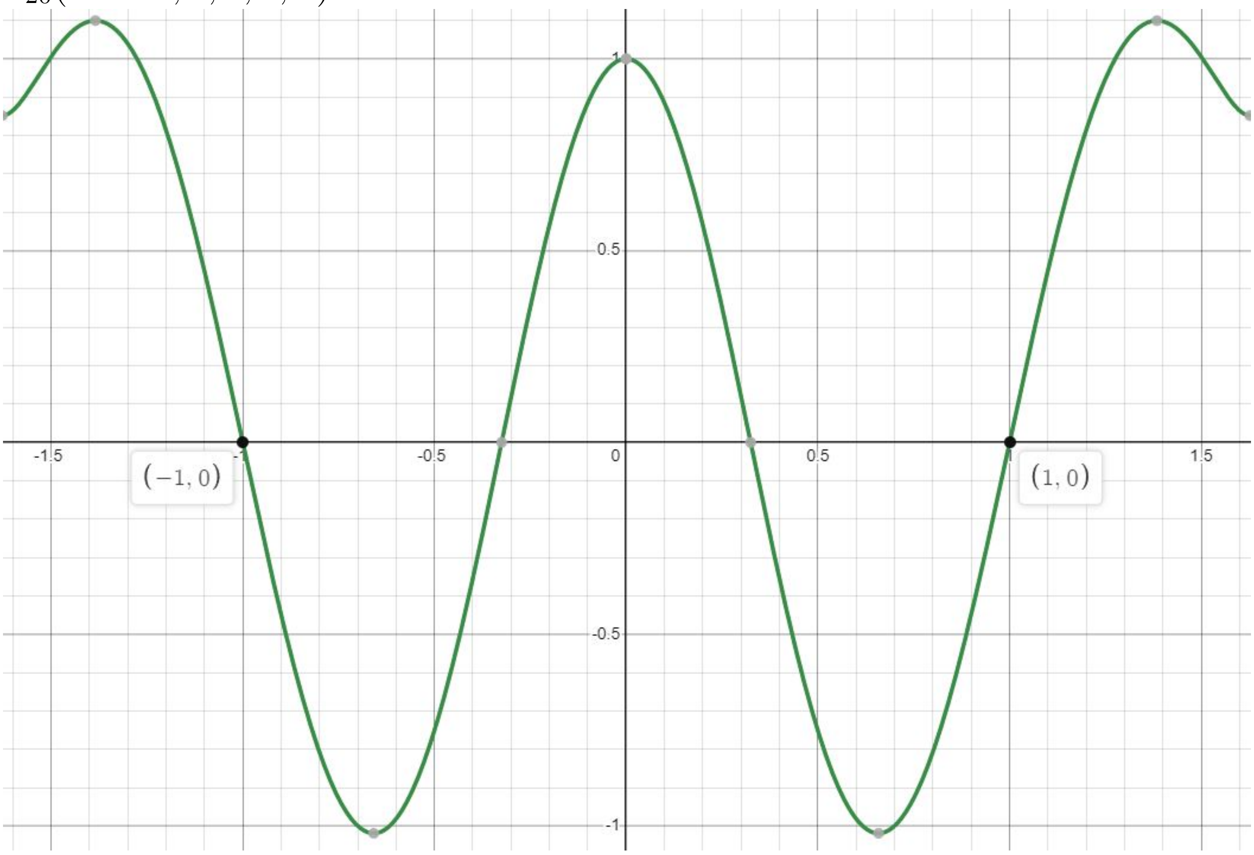
$T_{14}(2.972, 2, 1, 0, x)$:



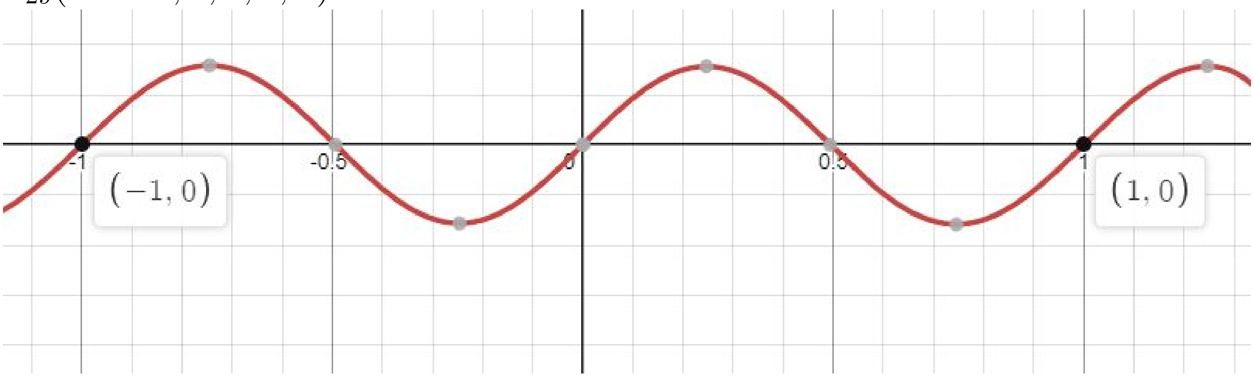
$T_{22}(10.983, 2, 1, 0, x)$:



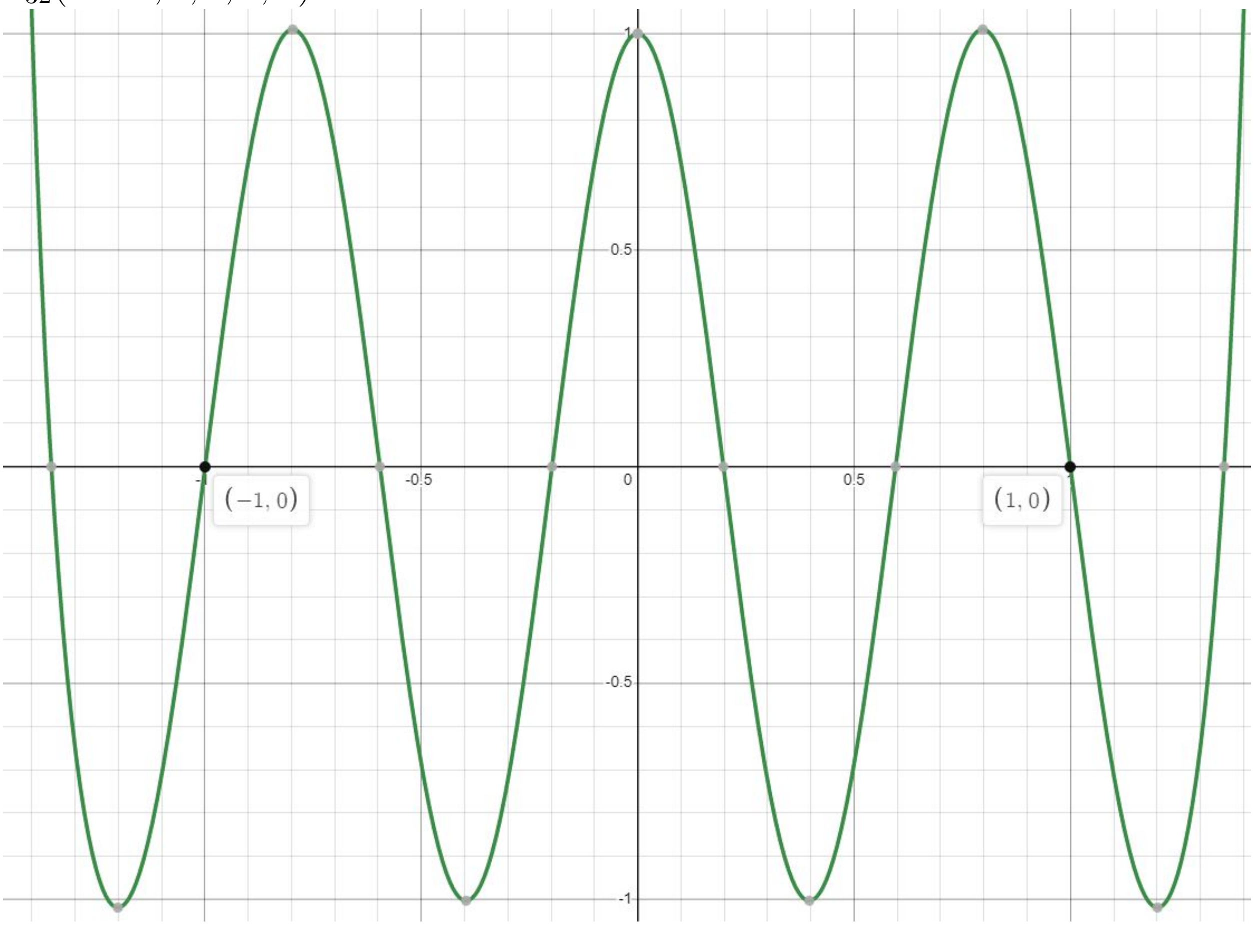
$T_{20}(23.453, 2, 1, 0, x)$:



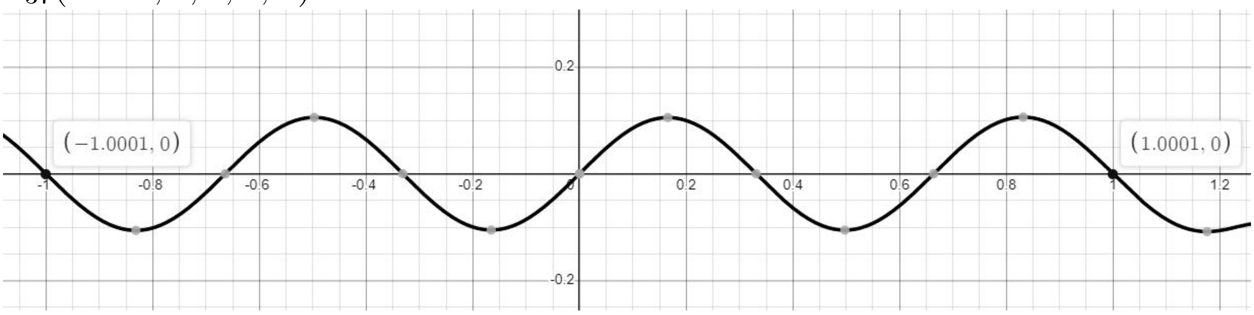
$T_{29}(40.766, 2, 0, 1, x)$:



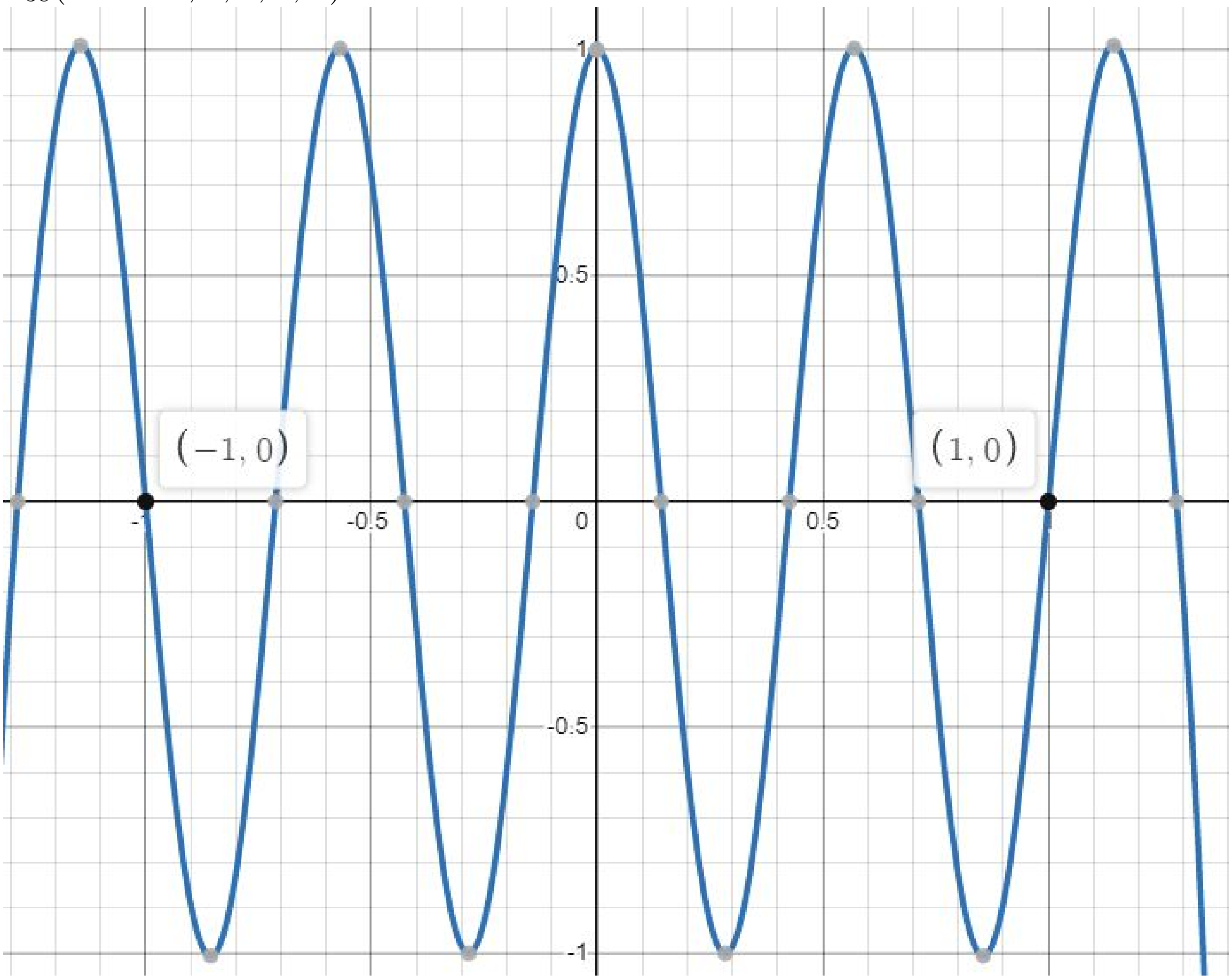
$T_{32}(62.99, 2, 1, 0, x)$:



$T_{37}(90.14, 2, 0, 1, x)$:

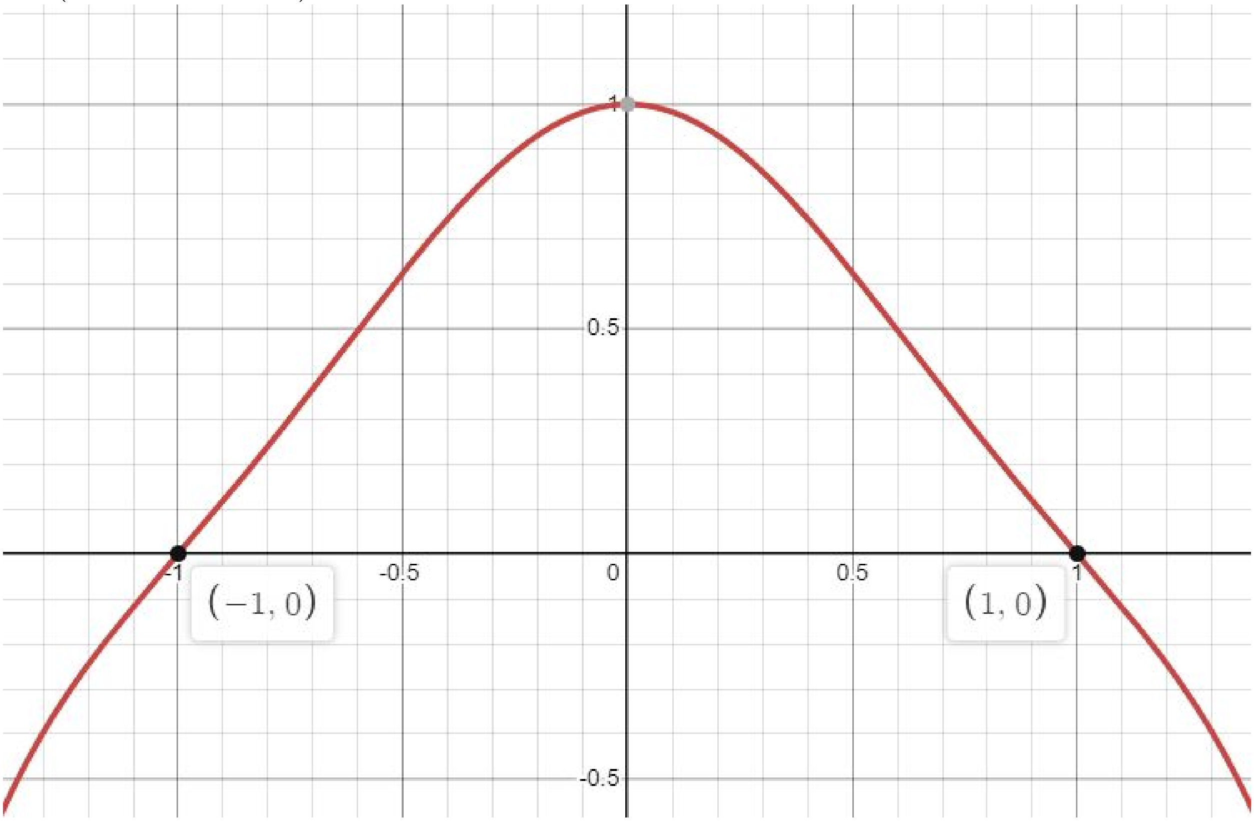


$T_{38}(122.222, 2, 1, 0, x)$:

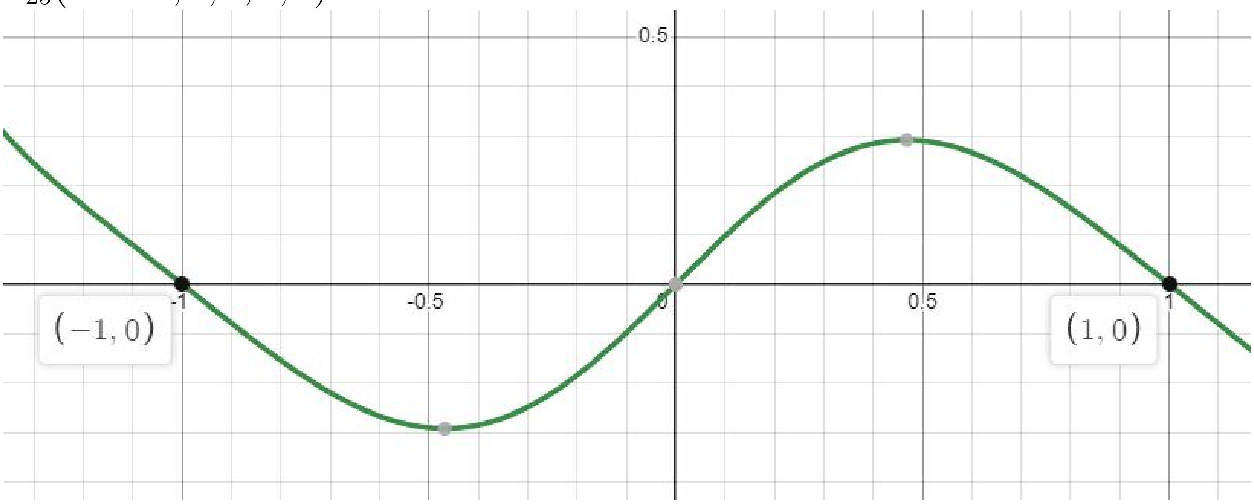


5.3 k=3

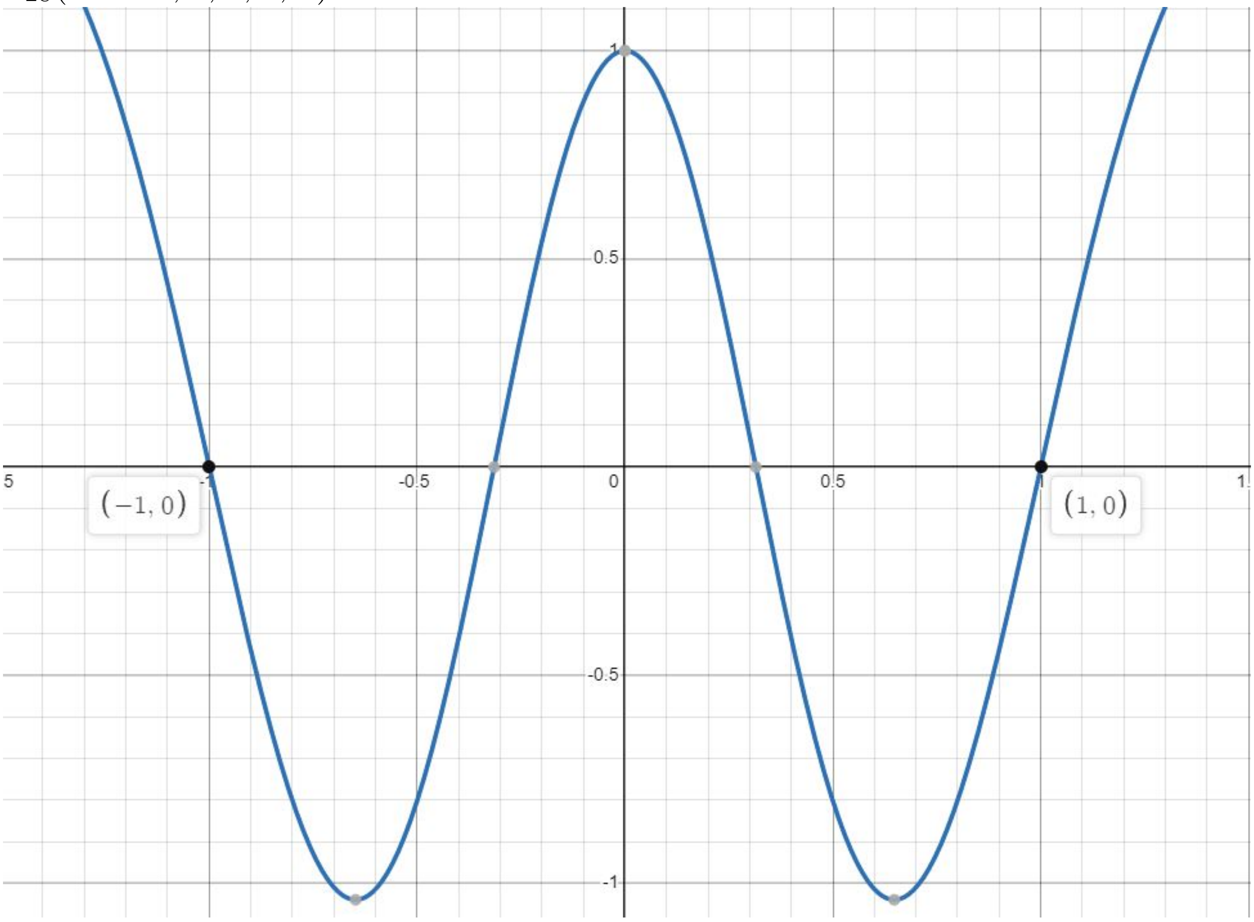
$T_{22}(3.556, 3, 1, 0, x)$:



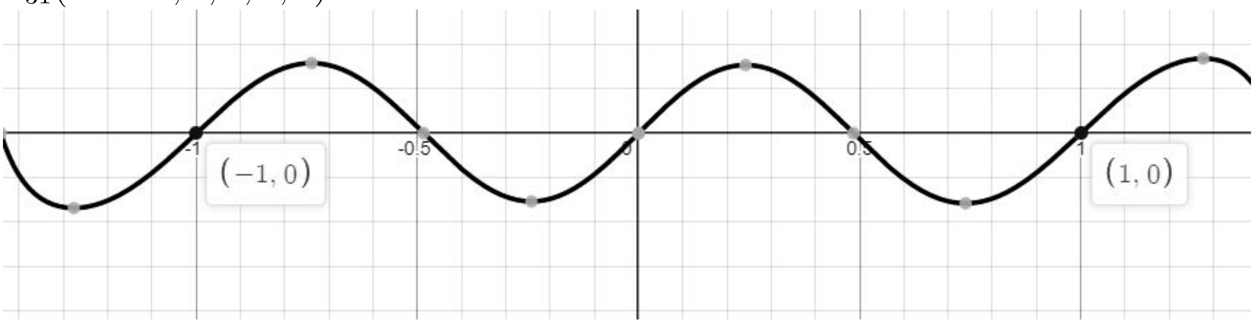
$T_{25}(12.325, 3, 0, 1, 1)$:



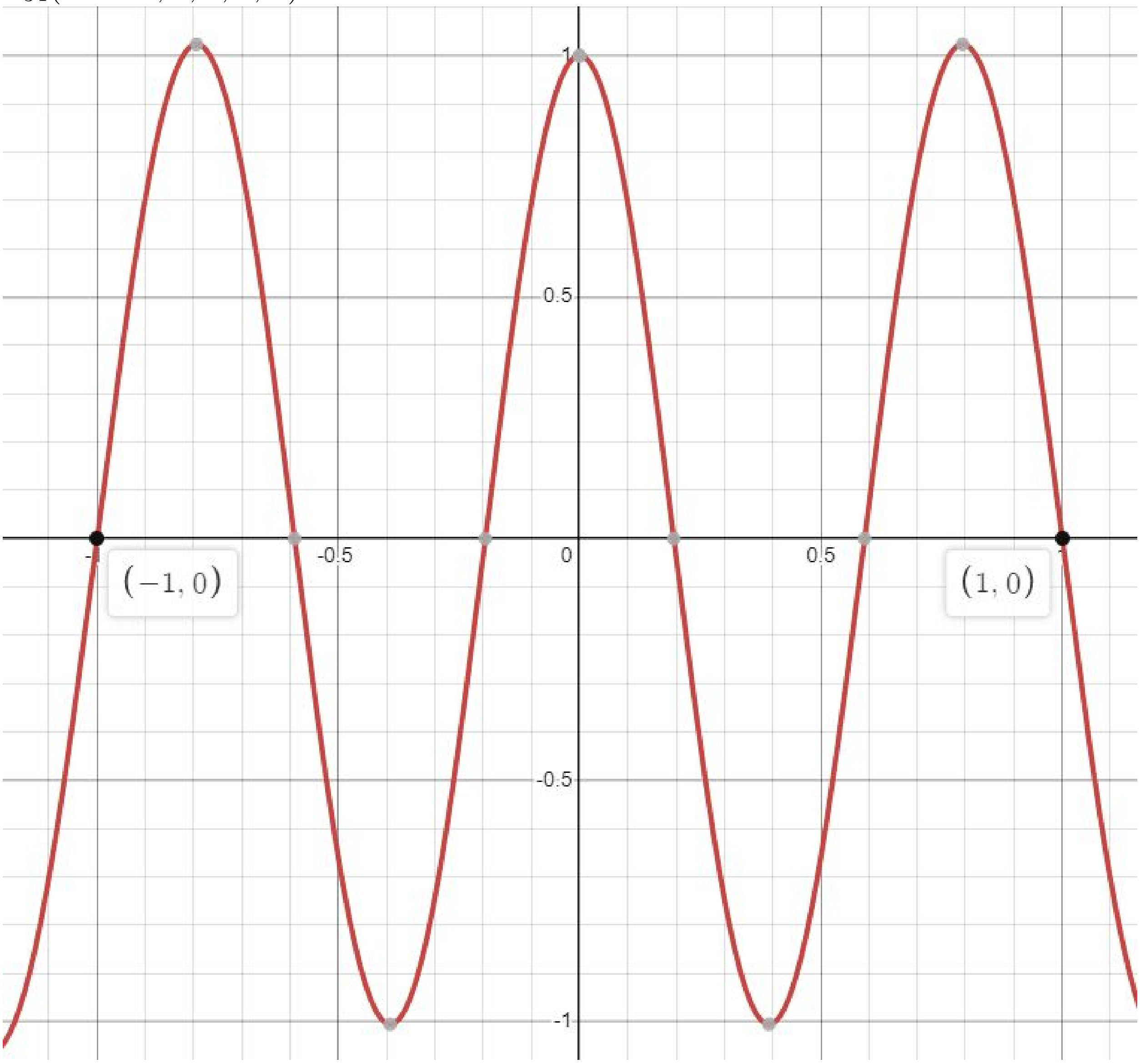
$T_{28}(25.015, 3, 1, 0, x)$:



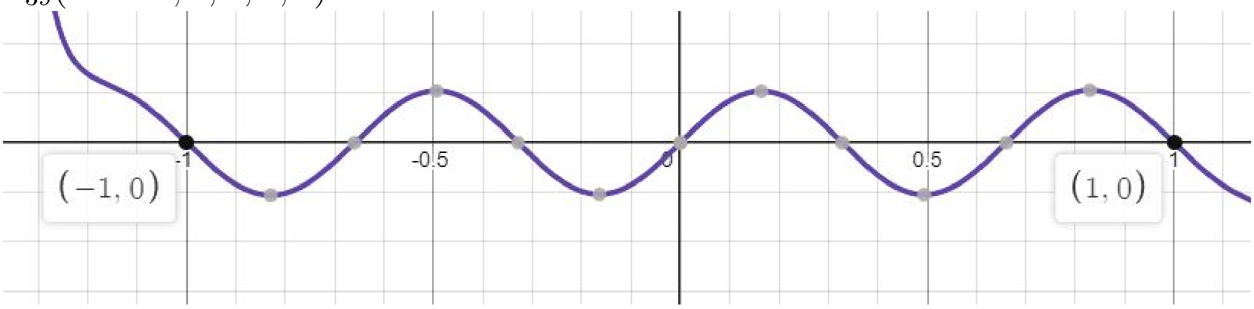
$T_{31}(42.388, 3, 0, 1, 1)$:



$T_{34}(64.632, 3, 1, 0, x)$:

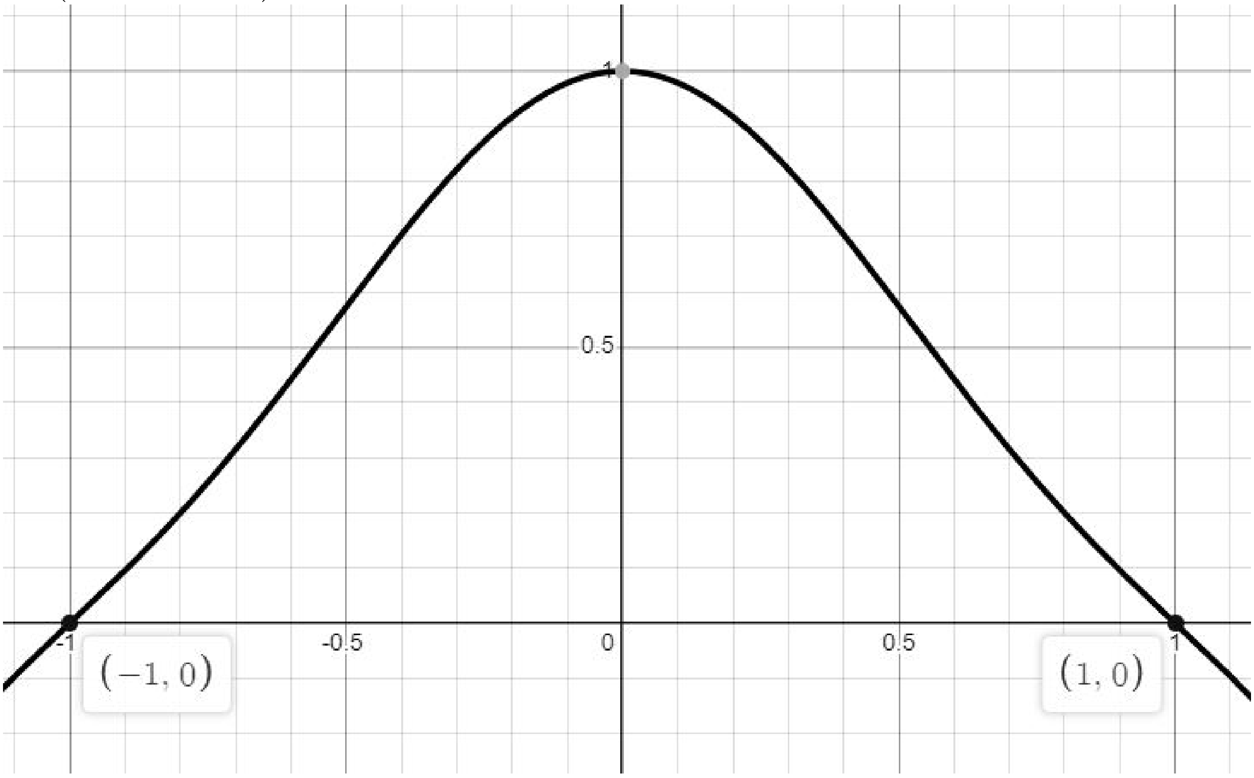


$T_{39}(91.792, 3, 0, 1, 1)$:

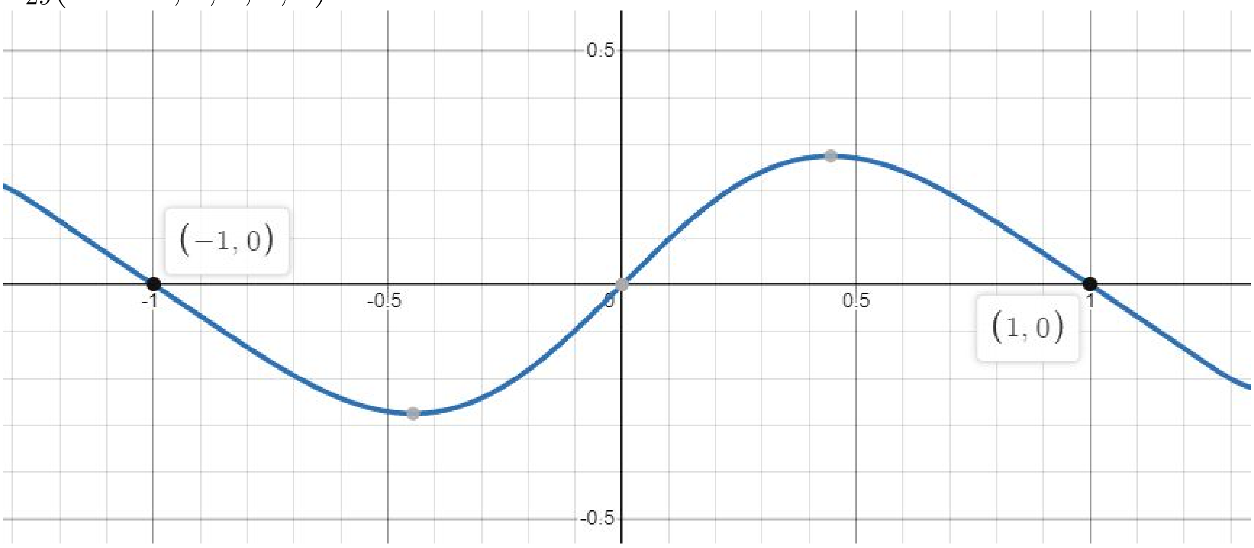


5.4 k=4

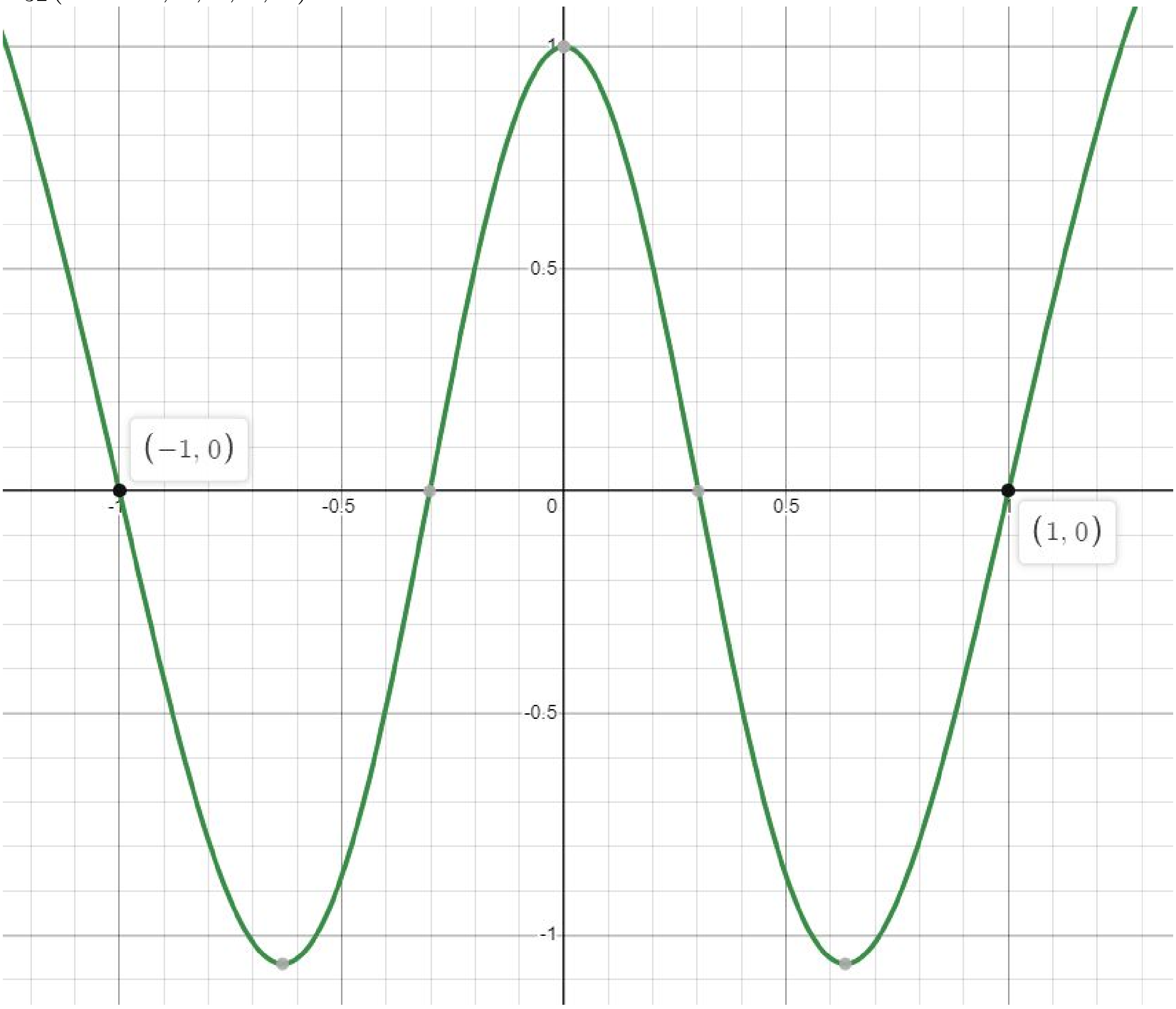
$T_{28}(4.3, 4, 1, 0, x)$:



$T_{29}(14.119, 4, 0, 1, 1)$:

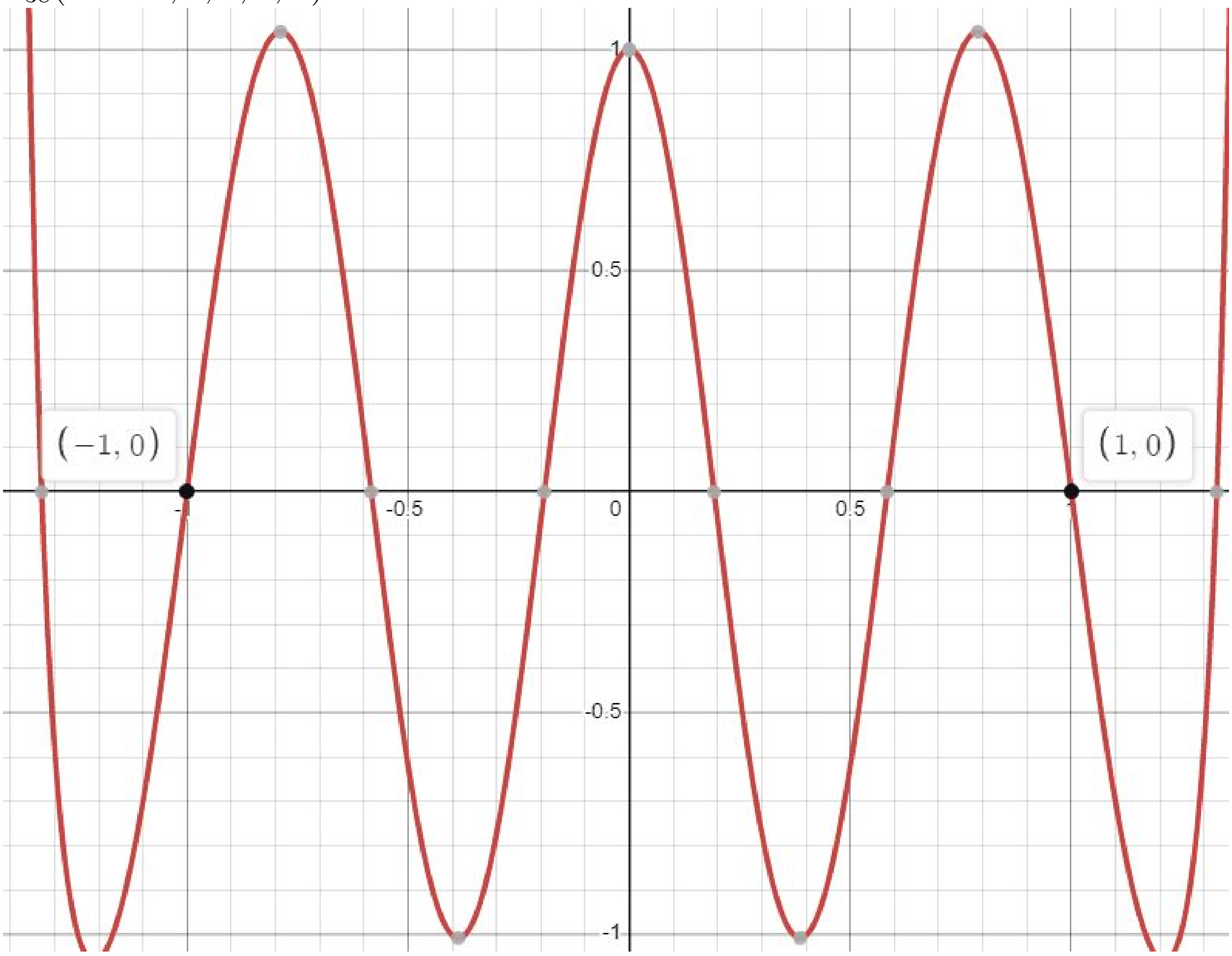


$T_{32}(27.198, 4, 1, 0, x)$:

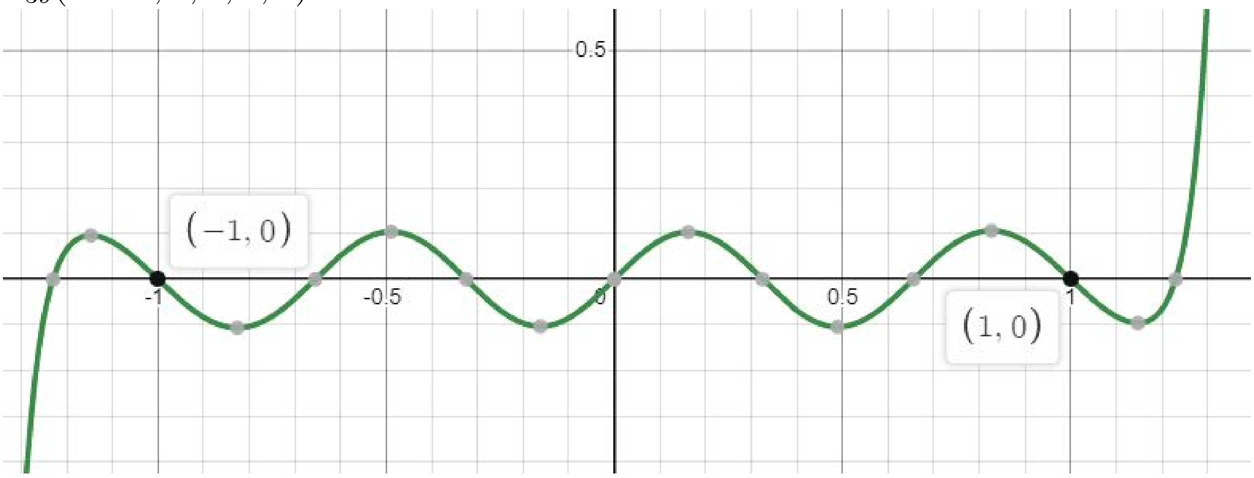


$T_{35}(44.677, 4, 0, 1, 1)$:

$T_{38}(66.951, 4, 1, 0, x)$:



$T_{39}(94.12, 4, 0, 1, 1)$:



6 Part 2

6.1 Goal of part 2

Our first next goal is the following:

Define $\lambda_{a,b} = a(a+1) - b^2$ and
$B_a = \{(l,m) : l \in \mathbb{N}_0, m \in \mathbb{Z}, |m| \leq l$ and $a = \lambda_{l,m}\}$.

We want to find the smallest positive integer n number such that $|B_n| \geq 15$ (if there exists one) and we want to get B_n.

Assuming n exists, if n is too large for our computer to handle such high values, we will try to find other smaller positive integers m_i such that B_{m_i} is as large as possible.

Once we have obtained B_n we will define $a_{l,m}, b_{l,m}$ as being two $N(0,1)$-random real numbers associated with each (l,m) couple.

Finally, we will graph $\sum_{(l,m)\in B_{l,m}}[a_{l,m}cos(my)P_l^m(cos(x))+b_{l,m}sin(my)P_l^m(cos(x))]$ where $P_l^m(cos(x))$ is the l,m-Legendre function. We will then try to graph the zero set of each of these graphs.

As said above, if some couple (l,m) is too large to be handled by our computer, we will have to decrease the value of n or only take a subset of B_n as our set.

6.2 Finding n

To find n, we will use a java program that will iterate through integers i from 0 to 10000 (we are hoping that $n \leq 10000$ for now).
For each of these integers i, we will count how many positive integers j where $|j| \leq i$ there are such that $\sqrt{j(j+1)-i}$ is an integer. This number of integers will be equal to $|B_i|$. Then as soon as we have $B_i \geq 15$, we have found $n = i$.

The reason why that works is because if $k = \pm\sqrt{j(j+1)-i} \in \mathbb{Z}$, then $|k| = \sqrt{j(j+1)-i} = \sqrt{j^2+j-i} \leq \sqrt{j^2} = j$, so $|k| < j$, $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.
Then $j(j+1)-k^2 = j(j+1)-(\sqrt{j(j+1)-i})^2 = j(j+1)-j(j+1)+i = i$.
That means $(j,k) \in B_i$.
So if we find at least 15 couples $(j,k) \in B_i$, then we have found that $n = i$.

Also, we don't need to verify for values of j greater than i, since if $j(j+1)-k^2 = i$, then it must be that $j \leq i$.

Proof:
Let $i \in \mathbb{N}_0$. Now, assume $\exists j \in \mathbb{N}_0, k \in \mathbb{Z}$, where $i, |k| < j$ such that $i = j(j+1)-k^2$.
We have $j^2 > k^2$ and $j > i$, hence $j(j+1) = j^2+j > i+k^2 = j(j+1)$.
This is a contradiction.
Q.E.D.

Here is another elegant proof:
Let $i \in \mathbb{N}_0$. Now, assume $\exists j \in \mathbb{N}_0, k \in \mathbb{Z}$, where $i, |k| < j$ such that $i = j(j+1)-k^2$.

Then let $a = (j-i) \in \mathbb{N}$, $j = a+i$. We have that $|k| = \sqrt{j(j+1)-i} = \sqrt{(a+i)(a+i+1)-i} = \sqrt{(a^2+ia+a+ia+i^2+i)-i} = \sqrt{a^2+2ia+i^2+a} = \sqrt{(a+i)^2+a} > \sqrt{(a+i)^2} = a+i$.

Also, $|k| = \sqrt{(a+i)^2+a} < \sqrt{(a+i)^2+a+a+2i+1} = \sqrt{(a+i)^2+2(a+i)+1} = \sqrt{((a+i)+1)^2} = a+i+1$.

So we have that $a+i < |k| < a+i+1$, which contradicts the fact that $k \in \mathbb{Z}$, so our initial assumption that $\exists j \in \mathbb{N}_0, k \in \mathbb{Z}$, where $i, |k| < j$ such that $i = j(j+1)-k^2$ was false.
Q.E.D.

Here is our code:

```
for (int i=0; i<10000; i++) {  // iterating from 0 to 10000

    int cnt = 0;  // counter

    for (int j=-i-1; j<=i; j++) {  // iterating from -i to i

        if (j*(j+1)-i >= 0) {  // to make sure we don't get the sqrt of a negative number
            if (Math.floor(Math.sqrt(j*(j+1)-i)) == Math.sqrt(j*(j+1)-i)) {  // Verifying if j*(j+1)-i is an integer
                cnt++;  // increases the counter by 1
            }
        }

    }

    if (cnt>=15) {  // breaks as soon as we find cnt is greater or equal to 15 for the current i

        System.out.print("n=" + i + " and |B_n|=" + cnt);  // outputs n and |B_n|
        break;

    }
}
```

And here is our output:

```
n=236 and |B_n|=16
```

Now that we know that $n = 236$ and that $|B_{236}| = 16$, we will find find the 16 couples in B_{236} with another program.

Here is our code:

```
for (int j=0; j<=i; j++) {  // iterates from 0 to i
    if (j*(j+1)-236 >= 0) {

        if (Math.floor(Math.sqrt(j*(j+1)-236)) == Math.sqrt(j*(j+1)-236)) {  // exact same principle as above

            arr[ctt]=j;  // inserts the current integer in the array

            ctt++;  // increases the position in the array by 1 for the next number that will be inserted

        }
    }
}

for (int j : arr) {  // prints the couples

    System.out.print(" ("+j + ","+ (int)Math.sqrt(j*(j+1)-maxI) + ") ");

}

for (int j : arr) {  // prints the couples

    System.out.print(" ("+j + ","+ (int)-Math.sqrt(j*(j+1)-maxI) + ") ");

}
```

And here is our output:

```
(15,2)  (16,6)  (19,12)  (28,24)  (35,32)  (48,46)  (79,78)  (236,236)

(15,-2)  (16,-6)  (19,-12)  (28,-24)  (35,-32)  (48,-46)  (79,-78)  (236,-236)
```

However, when we enter $P_l^m(cos(x))$ in Maple for each couple (l, m), we see that it works for all of them, except (236,236) and (236,-236) because of a lack of computational power.

In fact, we observe that it stops working around $l = 120$. To counter this problem, we have many options:

1) Find the integer $n < 120$ such that $|B_n|$ is maximised.

2) Stick with $n = 236$.

3) Find the integer n such that $|\{(l, m) : l \in \mathbb{N}_0,\ l < 120,\ m \in \mathbb{Z},\ |m| \leq l$ and $a = \lambda_{l,m}\}|$ is maximised.

We will try all 3 options and graph some samples with different coefficients chosen with $N(0, 1)$:

1) Here is our code to find the integer $n < 120$ such that $|B_n|$ is maximised:

```
int maxCNT=0;  // Those variables will be useful to keep track of which integer has the most corresponding couples
int maxI = 0;

int cnt = 0;
int i=0;

for (i=0; i<120; i++) {  // iterating from 0 to 10000

    cnt = 0;  // counter

    for (int j=0; j<=i; j++) {  // iterating from -i to i

        if (j*(j+1)-i >= 0) {  // to make sure we don't get the sqrt of a negative number
            if (Math.floor(Math.sqrt(j*(j+1)-i)) == Math.sqrt(j*(j+1)-i)) {  // Verifying if j*(j+1)-i is an integer

                if(j==0) {
                    cnt++;  // increases the counter by 1 if j=0
                }
                else {
                    cnt+=2;  // increases the counter by 2 if j>0
                }
            }
        }

        if (cnt>maxCNT) {
            maxCNT = cnt;
            maxI = i;
        }

    }
}

System.out.print("|B_n|=" + maxCNT + " and n=" + maxI + "       ");
```

And here is our output:

```
|B_n|=10 and n=101
```

Here are the couples we obtain:
(10,3) (13,9) (21,19) (34,33) (101,101) (10,-3) (13,-9) (21,-19) (34,-33) (101,-101)

3) Here is our code to find the integer n such that $|\{(l,m) : l \in \mathbb{N}_0,\, l < 120,$ $m \in \mathbb{Z},\, |m| \leq l$ and $a = \lambda_{l,m}\}|$ is maximised:

```
int maxCNT=0;  // Those variables will be useful to keep track of which integer has the most corresponding couples
int maxI = 0;

int cnt = 0;
int i=0;

for (i=0; i<=100000; i++) {  // iterating from 0 to 100000

    cnt = 0;  // counter

    for (int j=0; j<120; j++) {  // iterating from 0 to 119

        if (j*(j+1)-i >= 0) {  // to make sure we don't get the sqrt of a negative number
            if (Math.floor(Math.sqrt(j*(j+1)-i)) == Math.sqrt(j*(j+1)-i)) {  // Verifying if j*(j+1)-i is an integer

                if(j*(j+1)==i) {
                    cnt++;  // increases the counter by 1 if j=0
                }
                else {
                    cnt+=2;  // increases the counter by 2 if j>0
                }
            }
        }

        if (cnt>maxCNT) {
            maxCNT = cnt;
            maxI = i;
        }

    }
}
```

We get $n = 866$.

Here are the couples we obtain:
(29,2) (30,8) (33,16) (34,18) (46,36) (61,54) (81,76) (98,94)

(29,-2) (30,-8) (33,-16) (34,-18) (46,-36) (61,-54) (81,-76) (98,-94)

We now have 3 different values of n $(56, 101$ and $866)$ for which we will graph many samples.

6.3 Define the coefficients

By randomly choosing the coefficients with $N(0,1)$ we have obtained those lists of coefficients:

lambda = 101

#	Trig	l	m	1	2	3	4	5	6
1	sin	10	3	0.5377	1.8339	-2.2588	0.8622	0.3188	-1.3077
2	sin	13	9	-0.4336	0.3426	3.5784	2.7694	-1.3499	3.0349
3	sin	21	19	0.7254	-0.0631	0.7147	-0.205	-0.1241	1.4897
4	sin	34	33	1.409	1.4172	0.6715	-1.2075	0.7172	1.6302
5	sin	101	101	0.4889	1.0347	0.7269	-0.3034	0.2939	-0.7873
6	sin	10	-3	0.8884	-1.1471	-1.0689	-0.8095	-2.9443	1.4384
7	sin	13	-9	0.3252	-0.7549	1.3703	-1.7115	-0.1022	-0.2414
8	sin	21	-19	0.3192	0.3129	-0.8649	-0.0301	-0.1649	0.6277
9	sin	34	-33	1.0933	1.1093	-0.8637	0.0774	-1.2141	-1.1135
10	sin	101	-101	-0.0068	1.5326	-0.7697	0.3714	-0.2256	1.1174
11	cos	10	3	-1.0891	0.0326	0.5525	1.1006	1.5442	0.0859
12	cos	13	9	-1.4916	-0.7423	-1.0616	2.3505	-0.6156	0.7481
13	cos	21	19	-0.1924	0.8886	-0.7648	-1.4023	-1.4224	0.4882
14	cos	34	33	-0.1774	-0.1961	1.4193	0.2916	0.1978	1.5877
15	cos	101	101	-0.8045	0.6966	0.8351	-0.2437	0.2157	-1.1658
16	cos	10	-3	-1.148	0.1049	0.7223	2.5855	-0.6669	0.1873
17	cos	13	-9	-0.0825	-1.933	-0.439	-1.7947	0.8404	-0.888
18	cos	21	-19	0.1001	-0.5445	0.3035	-0.6003	0.49	0.7394
19	cos	34	-33	1.7119	-0.1941	-2.1384	-0.8396	1.3546	-1.0722
20	cos	101	-101	0.961	0.124	1.4367	-1.9609	-0.1977	-1.2078

lambda = 236

#	Trig	l	m	1	2	3	4	5	6
1	sin	15	2	0.5377	1.8339	-2.2588	0.8622	0.3188	-1.3077
2	sin	16	6	-0.4336	0.3426	3.5784	2.7694	-1.3499	3.0349
3	sin	19	12	0.7254	-0.0631	0.7147	-0.205	-0.1241	1.4897
4	sin	28	18	1.409	1.4172	0.6715	-1.2075	0.7172	1.6302
5	sin	35	24	0.4889	1.0347	0.7269	-0.3034	0.2939	-0.7873
6	sin	48	46	0.8884	-1.1471	-1.0689	-0.8095	-2.9443	1.4384
7	sin	79	78	0.3252	-0.7549	1.3703	-1.7115	-0.1022	-0.2414
8	sin	15	-2	0.3192	0.3129	-0.8649	-0.0301	-0.1649	0.6277
9	sin	16	-6	1.0933	1.1093	-0.8637	0.0774	-1.2141	-1.1135
10	sin	19	-12	-0.0068	1.5326	-0.7697	0.3714	-0.2256	1.1174
11	sin	28	-18	-1.0891	0.0326	0.5525	1.1006	1.5442	0.0859
12	sin	35	-24	-1.4916	-0.7423	-1.0616	2.3505	-0.6156	0.7481
13	sin	48	-46	-0.1924	0.8886	-0.7648	-1.4023	-1.4224	0.4882
14	sin	79	-78	-0.1774	-0.1961	1.4193	0.2916	0.1978	1.5877
15	cos	15	2	-0.8045	0.6966	0.8351	-0.2437	0.2157	-1.1658
16	cos	16	6	-1.148	0.1049	0.7223	2.5855	-0.6669	0.1873
17	cos	19	12	-0.0825	-1.933	-0.439	-1.7947	0.8404	-0.888
18	cos	28	18	0.1001	-0.5445	0.3035	-0.6003	0.49	0.7394
19	cos	35	24	1.7119	-0.1941	-2.1384	-0.8396	1.3546	-1.0722
20	cos	48	46	0.961	0.124	1.4367	-1.9609	-0.1977	-1.2078
21	cos	79	78	2.908	0.8252	1.379	-1.0582	-0.4686	-0.2725
22	cos	15	-2	1.0984	-0.2779	0.7015	-2.0518	-0.3538	-0.8236
23	cos	16	-6	-1.5771	0.508	0.282	0.0335	-1.3337	1.1275
24	cos	19	-12	0.3502	-0.2991	0.0229	0.262	-1.7502	-0.2857
25	cos	28	-18	-0.8314	-0.9792	-1.1564	-0.5336	-2.0026	0.9642
26	cos	35	-24	0.5201	-0.02	-0.0348	-0.7982	1.0187	-0.1332
27	cos	48	-46	-0.7145	1.3514	-0.2248	-0.589	-0.2938	-0.8479
28	cos	79	-78	-1.1201	2.526	1.6555	0.3075	-1.2571	-0.8655

lambda = 866

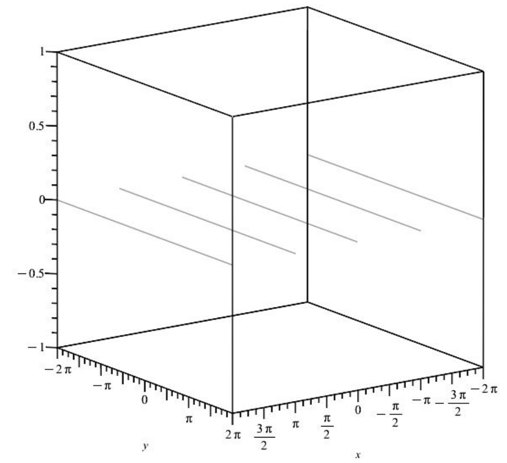
#	Trig	l	m	1	2	3	4	5	6
1	sin	29	2	0.5377	1.8339	-2.2588	0.8622	0.3188	-1.3077
2	sin	30	8	-0.4336	0.3426	3.5784	2.7694	-1.3499	3.0349
3	sin	33	16	0.7254	-0.0631	0.7147	-0.205	-0.1241	1.4897
4	sin	34	18	1.409	1.4172	0.6715	-1.2075	0.7172	1.6302
5	sin	46	36	0.4889	1.0347	0.7269	-0.3034	0.2939	-0.7873
6	sin	61	54	0.8884	-1.1471	-1.0689	-0.8095	-2.9443	1.4384
7	sin	81	76	0.3252	-0.7549	1.3703	-1.7115	-0.1022	-0.2414
8	sin	98	94	0.3192	0.3129	-0.8649	-0.0301	-0.1649	0.6277
9	sin	29	-2	1.0933	1.1093	-0.8637	0.0774	-1.2141	-1.1135
10	sin	30	-8	-0.0068	1.5326	-0.7697	0.3714	-0.2256	1.1174
11	sin	33	-16	-1.0891	0.0326	0.5525	1.1006	1.5442	0.0859
12	sin	34	-18	-1.4916	-0.7423	-1.0616	2.3505	-0.6156	0.7481
13	sin	46	-36	-0.1924	0.8886	-0.7648	-1.4023	-1.4224	0.4882
14	sin	61	-54	-0.1774	-0.1961	1.4193	0.2916	0.1978	1.5877
15	sin	81	-76	-0.8045	0.6966	0.8351	-0.2437	0.2157	-1.1658
16	sin	98	-94	-1.148	0.1049	0.7223	2.5855	-0.6669	0.1873
17	cos	29	2	-0.0825	-1.933	-0.439	-1.7947	0.8404	-0.888
18	cos	30	8	0.1001	-0.5445	0.3035	-0.6003	0.49	0.7394
19	cos	33	16	1.7119	-0.1941	-2.1384	-0.8396	1.3546	-1.0722
20	cos	34	18	0.961	0.124	1.4367	-1.9609	-0.1977	-1.2078
21	cos	46	36	2.908	0.8252	1.379	-1.0582	-0.4686	-0.2725
22	cos	61	54	1.0984	-0.2779	0.7015	-2.0518	-0.3538	-0.8236
23	cos	81	76	-1.5771	0.508	0.282	0.0335	-1.3337	1.1275
24	cos	98	94	0.3502	-0.2991	0.0229	0.262	-1.7502	-0.2857
25	cos	29	-2	-0.8314	-0.9792	-1.1564	-0.5336	-2.0026	0.9642
26	cos	30	-8	0.5201	-0.02	-0.0348	-0.7982	1.0187	-0.1332
27	cos	33	-16	-0.7145	1.3514	-0.2248	-0.589	-0.2938	-0.8479
28	cos	34	-18	-1.1201	2.526	1.6555	0.3075	-1.2571	-0.8655
29	cos	46	-36	-0.1765	0.7914	-1.332	-2.3299	-1.4491	0.3335
30	cos	61	-54	0.3914	0.4517	-0.1303	0.1837	-0.4762	0.862
31	cos	81	-76	-1.3617	0.455	-0.8487	-0.3349	0.5528	1.0391
32	cos	98	-94	-1.1176	1.2607	0.6601	-0.0679	-0.1952	-0.2176

At this point, we face a big problem for $\lambda_{m,l} = 101$: Maple cannot graph $P_l^m(cos(x))$ if m is odd, and all our corresponding couples have an odd value of m.

The reason why Maple is not able to graph for m odd is very unknown and we suspect that this is linked to the reason why this project has not been completed.

Here is what we obtain when we try to graph of of the samples for $\lambda_{m,l} = 101$:

```
plot3d((0.5377*sin(3y)*LegendreP(10, 3, cos(x))) + (-0.4336*sin(9y)*LegendreP(13, 9, cos(x))) + (0.7254*sin(19y)*LegendreP(21, 19, cos(x))) + (1.409*sin(33y)*LegendreP(34, 33, cos(x))) + (0.4889*sin(101y)
    *LegendreP(101, 101, cos(x))) + (0.8884*sin(-3y)*LegendreP(10,-3, cos(x))) + (0.3252*sin(-9y)*LegendreP(13,-9, cos(x))) + (0.3192*sin(-19y)*LegendreP(21,-19, cos(x))) + (1.0933*sin(-33y)
    *LegendreP(34,-33, cos(x))) + (-0.0068*sin(-101y)*LegendreP(101,-101, cos(x))) + (-1.0891*cos(3y)*LegendreP(10, 3, cos(x))) + (-1.4916*cos(9y)*LegendreP(13, 9, cos(x))) + (-0.1924*cos(19y)
    *LegendreP(21, 19, cos(x))) + (-0.1774*cos(33y)*LegendreP(34, 33, cos(x))) + (-0.8045*cos(101y)*LegendreP(101, 101, cos(x))) + (-1.148*cos(-3y)*LegendreP(10,-3, cos(x))) + (-0.0825*cos(-9y)
    *LegendreP(13,-9, cos(x))) + (0.1001*cos(-19y)*LegendreP(21,-19, cos(x))) + (1.7119*cos(-33y)*LegendreP(34,-33, cos(x))) + (0.961*cos(-101y)*LegendreP(101,-101, cos(x))))
```

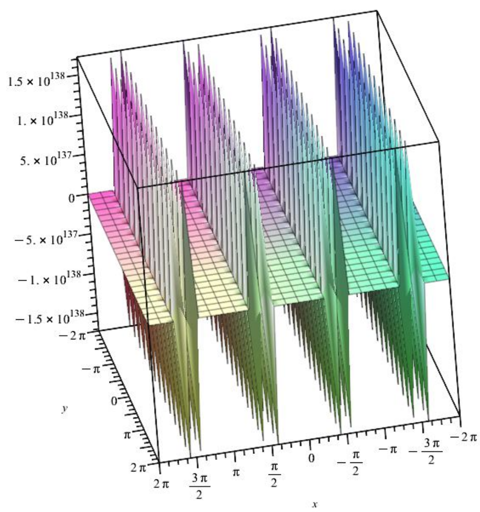


For this reason, we will only graph for $\lambda_{m,l} = 236$ and $\lambda_{m,l} = 866$.

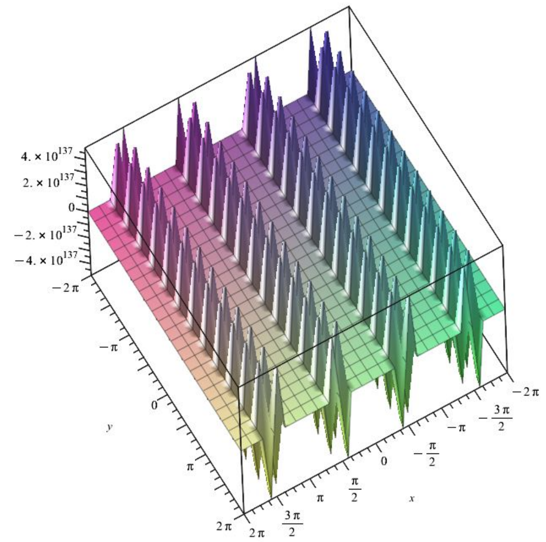
6.4 Graphing

$\lambda_{m,l} = 101$:

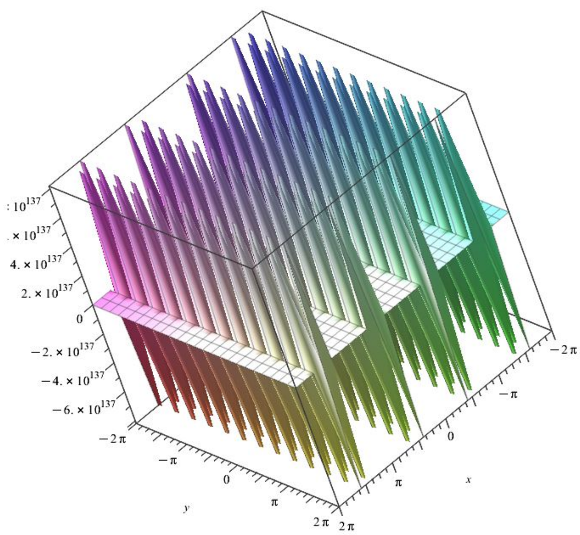
```
plot3d((-1.1201*cos(-78y)*LegendreP(79,-78, cos(x))) + (-0.7145*cos(-46y)*LegendreP(48,-46, cos(x))) + (0.5201*cos(-24y)*LegendreP(35,-24, cos(x))) + (-0.8314*cos(-18y)*LegendreP(28,-18,
    cos(x))) + (0.3502*cos(-12y)*LegendreP(19,-12, cos(x))) + (-1.5771*cos(-6y)*LegendreP(16,-6, cos(x))) + (1.0984*cos(-2y)*LegendreP(15,-2, cos(x))) + (2.908*cos(78y)*LegendreP(79, 78, cos(x)))
    + (0.961*cos(46y)*LegendreP(48, 46, cos(x))) + (1.7119*cos(24y)*LegendreP(35, 24, cos(x))) + (0.1001*cos(18y)*LegendreP(28, 18, cos(x))) + (-0.0825*cos(12y)*LegendreP(19, 12, cos(x))) + (-1.148
    *cos(6y)*LegendreP(16, 6, cos(x))) + (-0.8045*cos(2y)*LegendreP(15, 2, cos(x))) + (-0.1774*sin(-78y)*LegendreP(79,-78, cos(x))) + (-0.1924*sin(-46y)*LegendreP(48,-46, cos(x))) + (-1.4916*sin(
    -24y)*LegendreP(35,-24, cos(x))) + (-1.0891*sin(-18y)*LegendreP(28,-18, cos(x))) + (-0.0068*sin(-12y)*LegendreP(19,-12, cos(x))) + (1.0933*sin(-6y)*LegendreP(16,-6, cos(x))) + (0.3192*sin(
    -2y)*LegendreP(15,-2, cos(x))) + (0.3252*sin(78y)*LegendreP(79, 78, cos(x))) + (0.8884*sin(46y)*LegendreP(48, 46, cos(x))) + (0.4889*sin(24y)*LegendreP(35, 24, cos(x))) + (1.409*sin(18y)*LegendreP(28,
    18, cos(x))) + (0.7254*sin(12y)*LegendreP(19, 12, cos(x))) + (-0.4336*sin(6y)*LegendreP(16, 6, cos(x))) + (0.5377*sin(2y)*LegendreP(15, 2, cos(x))))
```



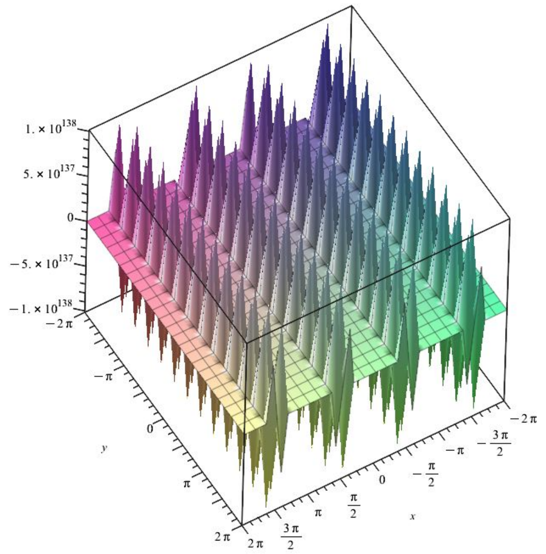
```
plot3d((2.526*cos(-78y)*LegendreP(79,-78, cos(x))) + (1.3514*cos(-46y)*LegendreP(48,-46, cos(x))) + (-0.02*cos(-24y)*LegendreP(35,-24, cos(x))) + (-0.9792*cos(-18y)*LegendreP(28,-18, cos(x)))
    + (-0.2991*cos(-12y)*LegendreP(19,-12, cos(x))) + (0.508*cos(-6y)*LegendreP(16,-6, cos(x))) + (-0.2779*cos(-2y)*LegendreP(15,-2, cos(x))) + (0.8252*cos(78y)*LegendreP(79, 78, cos(x))) + (0.124
    *cos(46y)*LegendreP(48, 46, cos(x))) + (-0.1941*cos(24y)*LegendreP(35, 24, cos(x))) + (-0.5445*cos(18y)*LegendreP(28, 18, cos(x))) + (-1.933*cos(12y)*LegendreP(19, 12, cos(x))) + (0.1049*cos(6y)
    *LegendreP(16, 6, cos(x))) + (0.6966*cos(2y)*LegendreP(15, 2, cos(x))) + (-0.1961*sin(-78y)*LegendreP(79,-78, cos(x))) + (0.8886*sin(-46y)*LegendreP(48,-46, cos(x))) + (-0.7423*sin(-24y)
    *LegendreP(35,-24, cos(x))) + (0.0326*sin(-18y)*LegendreP(28,-18, cos(x))) + (1.5326*sin(-12y)*LegendreP(19,-12, cos(x))) + (1.1093*sin(-6y)*LegendreP(16,-6, cos(x))) + (0.3129*sin(-2y)
    *LegendreP(15,-2, cos(x))) + (-0.7549*sin(78y)*LegendreP(79, 78, cos(x))) + (-1.1471*sin(46y)*LegendreP(48, 46, cos(x))) + (1.0347*sin(24y)*LegendreP(35, 24, cos(x))) + (1.4172*sin(18y)*LegendreP(28,
    18, cos(x))) + (-0.0631*sin(12y)*LegendreP(19, 12, cos(x))) + (0.3426*sin(6y)*LegendreP(16, 6, cos(x))) + (1.8339*sin(2y)*LegendreP(15, 2, cos(x))))
```



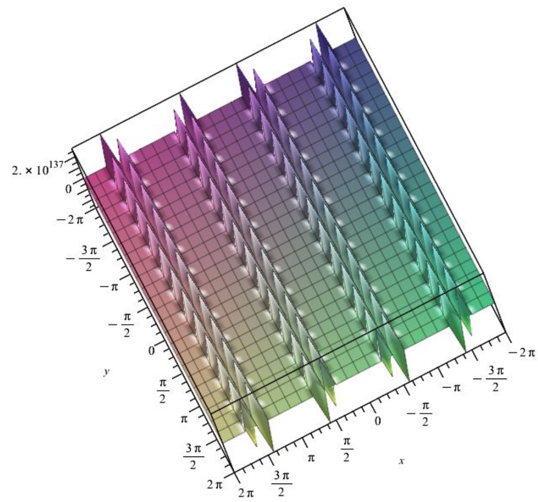

```
plot3d((1.6555*cos(-78y)*LegendreP(79,-78, cos(x))) + (-0.2248*cos(-46y)*LegendreP(48,-46, cos(x))) + (-0.0348*cos(-24y)*LegendreP(35,-24, cos(x))) + (-1.1564*cos(-18y)*LegendreP(28,-18,
    cos(x))) + (0.0229*cos(-12y)*LegendreP(19,-12, cos(x))) + (0.282*cos(-6y)*LegendreP(16,-6, cos(x))) + (0.7015*cos(-2y)*LegendreP(15,-2, cos(x))) + (1.379*cos(78y)*LegendreP(79, 78, cos(x)))
    + (1.4367*cos(46y)*LegendreP(48, 46, cos(x))) + (-2.1384*cos(24y)*LegendreP(35, 24, cos(x))) + (0.3035*cos(18y)*LegendreP(28, 18, cos(x))) + (-0.439*cos(12y)*LegendreP(19, 12, cos(x))) + (0.7223
    *cos(6y)*LegendreP(16, 6, cos(x))) + (0.8351*cos(2y)*LegendreP(15, 2, cos(x))) + (1.4193*sin(-78y)*LegendreP(79,-78, cos(x))) + (-0.7648*sin(-46y)*LegendreP(48,-46, cos(x))) + (-1.0616*sin(-24y)
    *LegendreP(35,-24, cos(x))) + (0.5525*sin(-18y)*LegendreP(28,-18, cos(x))) + (-0.7697*sin(-12y)*LegendreP(19,-12, cos(x))) + (-0.8637*sin(-6y)*LegendreP(16,-6, cos(x))) + (-0.8649*sin(-2y)
    *LegendreP(15,-2, cos(x))) + (1.3703*sin(78y)*LegendreP(79, 78, cos(x))) + (-1.0689*sin(46y)*LegendreP(48, 46, cos(x))) + (0.7269*sin(24y)*LegendreP(35, 24, cos(x))) + (0.6715*sin(18y)*LegendreP(28, 18,
    cos(x))) + (0.7147*sin(12y)*LegendreP(19, 12, cos(x))) + (3.5784*sin(6y)*LegendreP(16, 6, cos(x))) + (-2.2588*sin(2y)*LegendreP(15, 2, cos(x))))
```



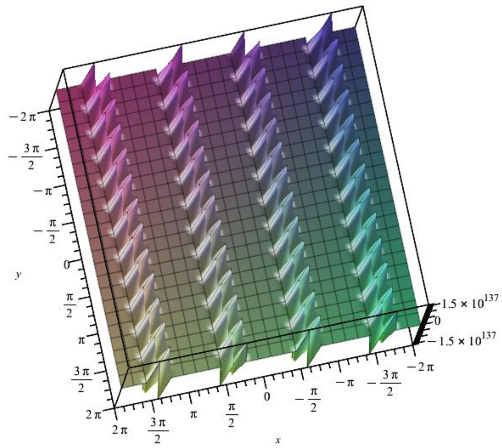
```
plot3d((0.3075*cos(-78y)*LegendreP(79,-78, cos(x))) + (-0.589*cos(-46y)*LegendreP(48,-46, cos(x))) + (-0.7982*cos(-24y)*LegendreP(35,-24, cos(x))) + (-0.5336*cos(-18y)*LegendreP(28,-18,
    cos(x))) + (0.262*cos(-12y)*LegendreP(19,-12, cos(x))) + (0.0335*cos(-6y)*LegendreP(16,-6, cos(x))) + (-2.0518*cos(-2y)*LegendreP(15,-2, cos(x))) + (-1.0582*cos(78y)*LegendreP(79, 78, cos(x)))
    + (-1.9609*cos(46y)*LegendreP(48, 46, cos(x))) + (-0.8396*cos(24y)*LegendreP(35, 24, cos(x))) + (-0.6003*cos(18y)*LegendreP(28, 18, cos(x))) + (-1.7947*cos(12y)*LegendreP(19, 12, cos(x))) + (2.5855
    *cos(6y)*LegendreP(16, 6, cos(x))) + (-0.2437*cos(2y)*LegendreP(15, 2, cos(x))) + (0.2916*sin(-78y)*LegendreP(79,-78, cos(x))) + (-1.4023*sin(-46y)*LegendreP(48,-46, cos(x))) + (2.3505*sin(-24y)
    *LegendreP(35,-24, cos(x))) + (1.1006*sin(-18y)*LegendreP(28,-18, cos(x))) + (0.3714*sin(-12y)*LegendreP(19,-12, cos(x))) + (0.0774*sin(-6y)*LegendreP(16,-6, cos(x))) + (-0.0301*sin(-2y)
    *LegendreP(15,-2, cos(x))) + (-1.7115*sin(78y)*LegendreP(79, 78, cos(x))) + (-0.8095*sin(46y)*LegendreP(48, 46, cos(x))) + (-0.3034*sin(24y)*LegendreP(35, 24, cos(x))) + (-1.2075*sin(18y)
    *LegendreP(28, 18, cos(x))) + (-0.205*sin(12y)*LegendreP(19, 12, cos(x))) + (2.7694*sin(6y)*LegendreP(16, 6, cos(x))) + (0.8622*sin(2y)*LegendreP(15, 2, cos(x))))
```




```
plot3d((-1.2571*cos(-78y)*LegendreP(79,-78, cos(x))) + (-0.2938*cos(-46y)*LegendreP(48,-46, cos(x))) + (1.0187*cos(-24y)*LegendreP(35,-24, cos(x))) + (-2.0026*cos(-18y)*LegendreP(28,-18,
    cos(x))) + (-1.7502*cos(-12y)*LegendreP(19,-12, cos(x))) + (-1.3337*cos(-6y)*LegendreP(16,-6, cos(x))) + (-0.3538*cos(-2y)*LegendreP(15,-2, cos(x))) + (-0.4686*cos(78y)*LegendreP(79, 78,
    cos(x))) + (-0.1977*cos(46y)*LegendreP(48, 46, cos(x))) + (1.3546*cos(24y)*LegendreP(35, 24, cos(x))) + (0.49*cos(18y)*LegendreP(28, 18, cos(x))) + (0.8404*cos(12y)*LegendreP(19, 12, cos(x))) + (-0.6669
    *cos(6y)*LegendreP(16, 6, cos(x))) + (0.2157*cos(2y)*LegendreP(15, 2, cos(x))) + (0.1978*sin(-78y)*LegendreP(79,-78, cos(x))) + (-1.4224*sin(-46y)*LegendreP(48,-46, cos(x))) + (-0.6156*sin(-24y)
    *LegendreP(35,-24, cos(x))) + (1.5442*sin(-18y)*LegendreP(28,-18, cos(x))) + (-0.2256*sin(-12y)*LegendreP(19,-12, cos(x))) + (-1.2141*sin(-6y)*LegendreP(16,-6, cos(x))) + (-0.1649*sin(-2y)
    *LegendreP(15,-2, cos(x))) + (-0.1022*sin(78y)*LegendreP(79, 78, cos(x))) + (-2.9443*sin(46y)*LegendreP(48, 46, cos(x))) + (0.2939*sin(24y)*LegendreP(35, 24, cos(x))) + (0.7172*sin(18y)*LegendreP(28,
    18, cos(x))) + (-0.1241*sin(12y)*LegendreP(19, 12, cos(x))) + (-1.3499*sin(6y)*LegendreP(16, 6, cos(x))) + (0.3188*sin(2y)*LegendreP(15, 2, cos(x))))
```

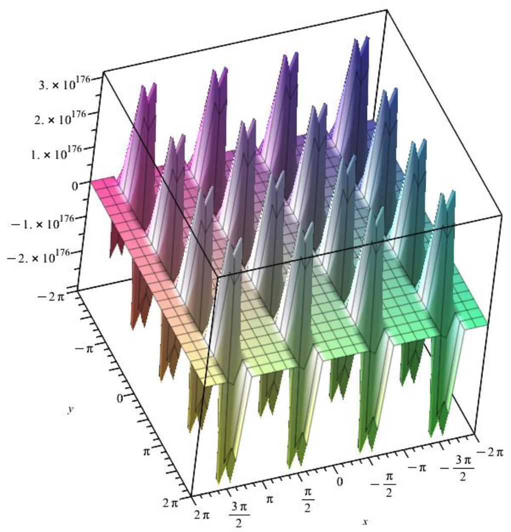


```
plot3d((-0.8655*cos(-78y)*LegendreP(79,-78, cos(x))) + (-0.8479*cos(-46y)*LegendreP(48,-46, cos(x))) + (-0.1332*cos(-24y)*LegendreP(35,-24, cos(x))) + (0.9642*cos(-18y)*LegendreP(28,-18,
    cos(x))) + (-0.2857*cos(-12y)*LegendreP(19,-12, cos(x))) + (1.1275*cos(-6y)*LegendreP(16,-6, cos(x))) + (-0.8236*cos(-2y)*LegendreP(15,-2, cos(x))) + (-0.2725*cos(78y)*LegendreP(79, 78,
    cos(x))) + (-1.2078*cos(46y)*LegendreP(48, 46, cos(x))) + (-1.0722*cos(24y)*LegendreP(35, 24, cos(x))) + (0.7394*cos(18y)*LegendreP(28, 18, cos(x))) + (-0.888*cos(12y)*LegendreP(19, 12, cos(x)))
    + (0.1873*cos(6y)*LegendreP(16, 6, cos(x))) + (-1.1658*cos(2y)*LegendreP(15, 2, cos(x))) + (1.5877*sin(-78y)*LegendreP(79,-78, cos(x))) + (0.4882*sin(-46y)*LegendreP(48,-46, cos(x))) + (0.7481*sin(
    -24y)*LegendreP(35,-24, cos(x))) + (0.0859*sin(-18y)*LegendreP(28,-18, cos(x))) + (1.1174*sin(-12y)*LegendreP(19,-12, cos(x))) + (-1.1135*sin(-6y)*LegendreP(16,-6, cos(x))) + (0.6277*sin(-2y)
    *LegendreP(15,-2, cos(x))) + (-0.2414*sin(78y)*LegendreP(79, 78, cos(x))) + (1.4384*sin(46y)*LegendreP(48, 46, cos(x))) + (-0.7873*sin(24y)*LegendreP(35, 24, cos(x))) + (1.6302*sin(18y)*LegendreP(28,
    18, cos(x))) + (1.4897*sin(12y)*LegendreP(19, 12, cos(x))) + (3.0349*sin(6y)*LegendreP(16, 6, cos(x))) + (-1.3077*sin(2y)*LegendreP(15, 2, cos(x))))
```

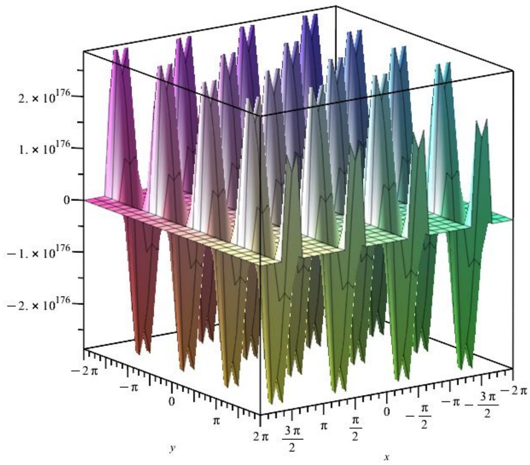


$\lambda_{m,l} = 866$:

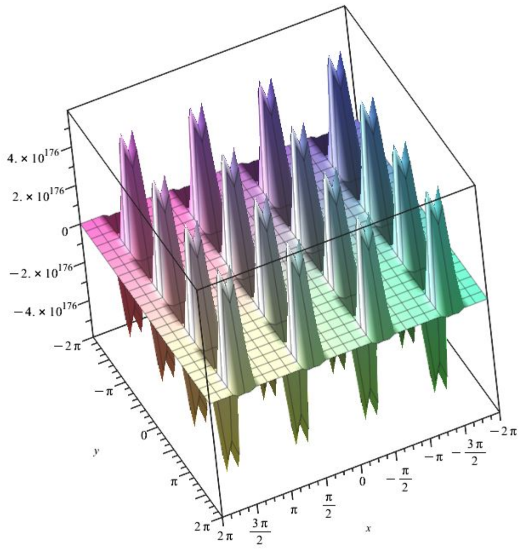
```
plot3d((-1.1176*cos(-94y)*LegendreP(98,-94, cos(x))) + (-1.3617*cos(-76y)*LegendreP(81,-76, cos(x))) + (0.3914*cos(-54y)*LegendreP(61,-54, cos(x))) + (-0.1765*cos(-36y)*LegendreP(46,-36,
    cos(x))) + (-1.1201*cos(-18y)*LegendreP(34,-18, cos(x))) + (-0.7145*cos(-16y)*LegendreP(33,-16, cos(x))) + (0.5201*cos(-8y)*LegendreP(30,-8, cos(x))) + (-0.8314*cos(-2y)*LegendreP(29,-2,
    cos(x))) + (0.3502*cos(94y)*LegendreP(98, 94, cos(x))) + (-1.5771*cos(76y)*LegendreP(81, 76, cos(x))) + (1.0984*cos(54y)*LegendreP(61, 54, cos(x))) + (2.908*cos(36y)*LegendreP(46, 36, cos(x))) + (0.961
    *cos(18y)*LegendreP(34, 18, cos(x))) + (1.7119*cos(16y)*LegendreP(33, 16, cos(x))) + (0.1001*cos(8y)*LegendreP(30, 8, cos(x))) + (-0.0825*cos(2y)*LegendreP(29, 2, cos(x))) + (-1.148*sin(-94y)
    *LegendreP(98,-94, cos(x))) + (-0.8045*sin(-76y)*LegendreP(81,-76, cos(x))) + (-0.1774*sin(-54y)*LegendreP(61,-54, cos(x))) + (-0.1924*sin(-36y)*LegendreP(46,-36, cos(x))) + (-1.4916*sin(
    -18y)*LegendreP(34,-18, cos(x))) + (-1.0891*sin(-16y)*LegendreP(33,-16, cos(x))) + (-0.0068*sin(-8y)*LegendreP(30,-8, cos(x))) + (1.0933*sin(-2y)*LegendreP(29,-2, cos(x))) + (0.3192*sin(94y)
    *LegendreP(98, 94, cos(x))) + (0.3252*sin(76y)*LegendreP(81, 76, cos(x))) + (0.8884*sin(54y)*LegendreP(61, 54, cos(x))) + (0.4889*sin(36y)*LegendreP(46, 36, cos(x))) + (1.409*sin(18y)*LegendreP(34, 18,
    cos(x))) + (0.7254*sin(16y)*LegendreP(33, 16, cos(x))) + (-0.4336*sin(8y)*LegendreP(30, 8, cos(x))) + (0.5377*sin(2y)*LegendreP(29, 2, cos(x))))
```



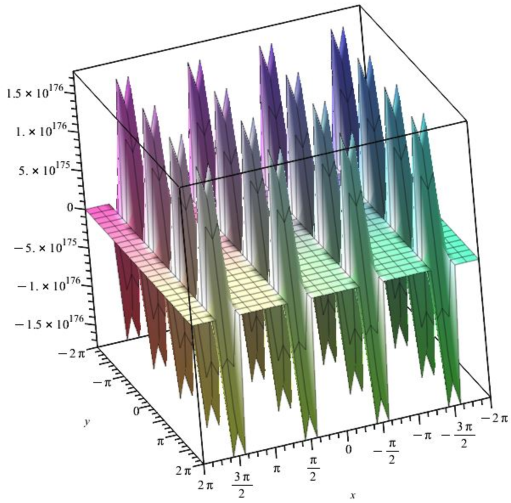
```
plot3d((1.2607*cos(−94y)*LegendreP(98,−94, cos(x))) + (0.455*cos(−76y)*LegendreP(81,−76, cos(x))) + (0.4517*cos(−54y)*LegendreP(61,−54, cos(x))) + (0.7914*cos(−36y)*LegendreP(46,−36, cos(x)))
    + (2.526*cos(−18y)*LegendreP(34,−18, cos(x))) + (1.3514*cos(−16y)*LegendreP(33,−16, cos(x))) + (−0.02*cos(−8y)*LegendreP(30,−8, cos(x))) + (−0.9792*cos(−2y)*LegendreP(29,−2, cos(x))) + (
    −0.2991*cos(94y)*LegendreP(98, 94, cos(x))) + (0.508*cos(76y)*LegendreP(81, 76, cos(x))) + (−0.2779*cos(54y)*LegendreP(61, 54, cos(x))) + (0.8252*cos(36y)*LegendreP(46, 36, cos(x))) + (0.124*cos(18y)
    *LegendreP(34, 18, cos(x))) + (−0.1941*cos(16y)*LegendreP(33, 16, cos(x))) + (−0.5445*cos(8y)*LegendreP(30, 8, cos(x))) + (−1.933*cos(2y)*LegendreP(29, 2, cos(x))) + (0.1049*sin(−94y)*LegendreP(98,
    −94, cos(x))) + (0.6966*sin(−76y)*LegendreP(81,−76, cos(x))) + (−0.1961*sin(−54y)*LegendreP(61,−54, cos(x))) + (0.8886*sin(−36y)*LegendreP(46,−36, cos(x))) + (−0.7423*sin(−18y)*LegendreP(34,
    −18, cos(x))) + (0.0326*sin(−16y)*LegendreP(33,−16, cos(x))) + (1.5326*sin(−8y)*LegendreP(30,−8, cos(x))) + (1.1093*sin(−2y)*LegendreP(29,−2, cos(x))) + (0.3129*sin(94y)*LegendreP(98, 94, cos(x)))
    + (−0.7549*sin(76y)*LegendreP(81, 76, cos(x))) + (−1.1471*sin(54y)*LegendreP(61, 54, cos(x))) + (1.0347*sin(36y)*LegendreP(46, 36, cos(x))) + (1.4172*sin(18y)*LegendreP(34, 18, cos(x))) + (−0.0631
    *sin(16y)*LegendreP(33, 16, cos(x))) + (0.3426*sin(8y)*LegendreP(30, 8, cos(x))) + (1.8339*sin(2y)*LegendreP(29, 2, cos(x))))
```



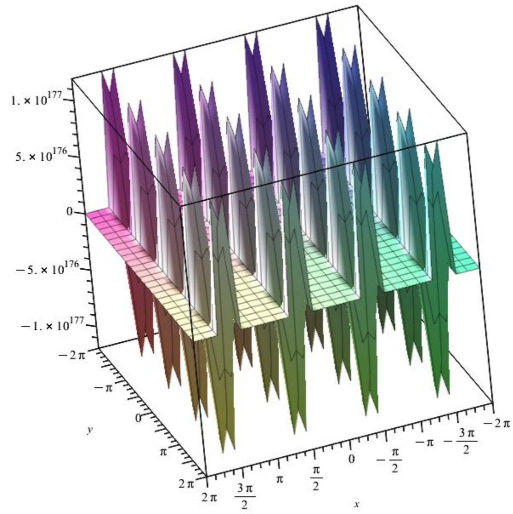

```
plot3d((0.6601*cos(−94y)*LegendreP(98,−94, cos(x))) + (−0.8487*cos(−76y)*LegendreP(81,−76, cos(x))) + (−0.1303*cos(−54y)*LegendreP(61,−54, cos(x))) + (−1.332*cos(−36y)*LegendreP(46,−36,
    cos(x))) + (1.6555*cos(−18y)*LegendreP(34,−18, cos(x))) + (−0.2248*cos(−16y)*LegendreP(33,−16, cos(x))) + (−0.0348*cos(−8y)*LegendreP(30,−8, cos(x))) + (−1.1564*cos(−2y)*LegendreP(29,−2,
    cos(x))) + (0.0229*cos(94y)*LegendreP(98, 94, cos(x))) + (0.282*cos(76y)*LegendreP(81, 76, cos(x))) + (0.7015*cos(54y)*LegendreP(61, 54, cos(x))) + (1.379*cos(36y)*LegendreP(46, 36, cos(x))) + (1.4367
    *cos(18y)*LegendreP(34, 18, cos(x))) + (−2.1384*cos(16y)*LegendreP(33, 16, cos(x))) + (0.3035*cos(8y)*LegendreP(30, 8, cos(x))) + (−0.439*cos(2y)*LegendreP(29, 2, cos(x))) + (0.7223*sin(−94y)
    *LegendreP(98,−94, cos(x))) + (0.8351*sin(−76y)*LegendreP(81,−76, cos(x))) + (1.4193*sin(−54y)*LegendreP(61,−54, cos(x))) + (−0.7648*sin(−36y)*LegendreP(46,−36, cos(x))) + (−1.0616*sin(−18y)
    *LegendreP(34,−18, cos(x))) + (0.5525*sin(−16y)*LegendreP(33,−16, cos(x))) + (−0.7697*sin(−8y)*LegendreP(30,−8, cos(x))) + (−0.8637*sin(−2y)*LegendreP(29,−2, cos(x))) + (−0.8649*sin(94y)
    *LegendreP(98, 94, cos(x))) + (1.3703*sin(76y)*LegendreP(81, 76, cos(x))) + (−1.0689*sin(54y)*LegendreP(61, 54, cos(x))) + (0.7269*sin(36y)*LegendreP(46, 36, cos(x))) + (0.6715*sin(18y)*LegendreP(34, 18,
    cos(x))) + (0.7147*sin(16y)*LegendreP(33, 16, cos(x))) + (3.5784*sin(8y)*LegendreP(30, 8, cos(x))) + (−2.2588*sin(2y)*LegendreP(29, 2, cos(x))))
```



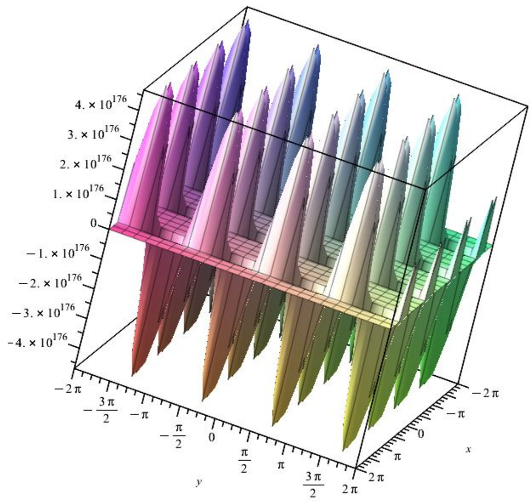
```
plot3d((-0.0679*cos(-94 y)*LegendreP(98,-94, cos(x))) + (-0.3349*cos(-76 y)*LegendreP(81,-76, cos(x))) + (0.1837*cos(-54 y)*LegendreP(61,-54, cos(x))) + (-2.3299*cos(-36 y)*LegendreP(46,-36,
    cos(x))) + (0.3075*cos(-18 y)*LegendreP(34,-18, cos(x))) + (-0.589*cos(-16 y)*LegendreP(33,-16, cos(x))) + (-0.7982*cos(-8 y)*LegendreP(30,-8, cos(x))) + (-0.5336*cos(-2 y)*LegendreP(29,-2,
    cos(x))) + (0.262*cos(94 y)*LegendreP(98, 94, cos(x))) + (0.0335*cos(76 y)*LegendreP(81, 76, cos(x))) + (-2.0518*cos(54 y)*LegendreP(61, 54, cos(x))) + (-1.0582*cos(36 y)*LegendreP(46, 36, cos(x))) + (
    -1.9609*cos(18 y)*LegendreP(34, 18, cos(x))) + (-0.8396*cos(16 y)*LegendreP(33, 16, cos(x))) + (-0.6003*cos(8 y)*LegendreP(30, 8, cos(x))) + (-1.7947*cos(2 y)*LegendreP(29, 2, cos(x))) + (2.5855*sin(
    -94 y)*LegendreP(98,-94, cos(x))) + (-0.2437*sin(-76 y)*LegendreP(81,-76, cos(x))) + (0.2916*sin(-54 y)*LegendreP(61,-54, cos(x))) + (-1.4023*sin(-36 y)*LegendreP(46,-36, cos(x))) + (2.3505*sin(
    -18 y)*LegendreP(34,-18, cos(x))) + (1.1006*sin(-16 y)*LegendreP(33,-16, cos(x))) + (0.3714*sin(-8 y)*LegendreP(30,-8, cos(x))) + (0.0774*sin(-2 y)*LegendreP(29,-2, cos(x))) + (-0.0301*sin(94 y)
    *LegendreP(98, 94, cos(x))) + (-1.7115*sin(76 y)*LegendreP(81, 76, cos(x))) + (-0.8095*sin(54 y)*LegendreP(61, 54, cos(x))) + (-0.3034*sin(36 y)*LegendreP(46, 36, cos(x))) + (-1.2075*sin(18 y)
    *LegendreP(34, 18, cos(x))) + (-0.205*sin(16 y)*LegendreP(33, 16, cos(x))) + (2.7694*sin(8 y)*LegendreP(30, 8, cos(x))) + (0.8622*sin(2 y)*LegendreP(29, 2, cos(x))))
```




```
plot3d((-0.1952*cos(-94 y)*LegendreP(98,-94, cos(x))) + (0.5528*cos(-76 y)*LegendreP(81,-76, cos(x))) + (-0.4762*cos(-54 y)*LegendreP(61,-54, cos(x))) + (-1.4491*cos(-36 y)*LegendreP(46,-36,
    cos(x))) + (-1.2571*cos(-18 y)*LegendreP(34,-18, cos(x))) + (-0.2938*cos(-16 y)*LegendreP(33,-16, cos(x))) + (1.0187*cos(-8 y)*LegendreP(30,-8, cos(x))) + (-2.0026*cos(-2 y)*LegendreP(29,-2,
    cos(x))) + (-1.7502*cos(94 y)*LegendreP(98, 94, cos(x))) + (-1.3337*cos(76 y)*LegendreP(81, 76, cos(x))) + (-0.3538*cos(54 y)*LegendreP(61, 54, cos(x))) + (-0.4686*cos(36 y)*LegendreP(46, 36, cos(x)))
    + (-0.1977*cos(18 y)*LegendreP(34, 18, cos(x))) + (1.3546*cos(16 y)*LegendreP(33, 16, cos(x))) + (0.49*cos(8 y)*LegendreP(30, 8, cos(x))) + (0.8404*cos(2 y)*LegendreP(29, 2, cos(x))) + (-0.6669*sin(-94 y)
    *LegendreP(98,-94, cos(x))) + (0.2157*sin(-76 y)*LegendreP(81,-76, cos(x))) + (0.1978*sin(-54 y)*LegendreP(61,-54, cos(x))) + (-1.4224*sin(-36 y)*LegendreP(46,-36, cos(x))) + (-0.6156*sin(-18 y)
    *LegendreP(34,-18, cos(x))) + (1.5442*sin(-16 y)*LegendreP(33,-16, cos(x))) + (-0.2256*sin(-8 y)*LegendreP(30,-8, cos(x))) + (-1.2141*sin(-2 y)*LegendreP(29,-2, cos(x))) + (-0.1649*sin(94 y)
    *LegendreP(98, 94, cos(x))) + (-0.1022*sin(76 y)*LegendreP(81, 76, cos(x))) + (-2.9443*sin(54 y)*LegendreP(61, 54, cos(x))) + (0.2939*sin(36 y)*LegendreP(46, 36, cos(x))) + (0.7172*sin(18 y)*LegendreP(34,
    18, cos(x))) + (-0.1241*sin(16 y)*LegendreP(33, 16, cos(x))) + (-1.3499*sin(8 y)*LegendreP(30, 8, cos(x))) + (0.3188*sin(2 y)*LegendreP(29, 2, cos(x))))
```



```
plot3d((−0.2176*cos(−94y)*LegendreP(98,−94, cos(x))) + (1.0391*cos(−76y)*LegendreP(81,−76, cos(x))) + (0.862*cos(−54y)*LegendreP(61,−54, cos(x))) + (0.3335*cos(−36y)*LegendreP(46,−36, cos(x)))
    + (−0.8655*cos(−18y)*LegendreP(34,−18, cos(x))) + (−0.8479*cos(−16y)*LegendreP(33,−16, cos(x))) + (−0.1332*cos(−8y)*LegendreP(30,−8, cos(x))) + (0.9642*cos(−2y)*LegendreP(29,−2, cos(x)))
    + (−0.2857*cos(94y)*LegendreP(98, 94, cos(x))) + (1.1275*cos(76y)*LegendreP(81, 76, cos(x))) + (−0.8236*cos(54y)*LegendreP(61, 54, cos(x))) + (−0.2725*cos(36y)*LegendreP(46, 36, cos(x))) + (−1.2078
    *cos(18y)*LegendreP(34, 18, cos(x))) + (−1.0722*cos(16y)*LegendreP(33, 16, cos(x))) + (0.7394*cos(8y)*LegendreP(30, 8, cos(x))) + (−0.888*cos(2y)*LegendreP(29, 2, cos(x))) + (0.1873*sin(−94y)
    *LegendreP(98,−94, cos(x))) + (−1.1658*sin(−76y)*LegendreP(81,−76, cos(x))) + (1.5877*sin(−54y)*LegendreP(61,−54, cos(x))) + (0.4882*sin(−36y)*LegendreP(46,−36, cos(x))) + (0.7481*sin(−18y)
    *LegendreP(34,−18, cos(x))) + (0.0859*sin(−16y)*LegendreP(33,−16, cos(x))) + (1.1174*sin(−8y)*LegendreP(30,−8, cos(x))) + (−1.1135*sin(−2y)*LegendreP(29,−2, cos(x))) + (0.6277*sin(94y)
    *LegendreP(98, 94, cos(x))) + (−0.2414*sin(76y)*LegendreP(81, 76, cos(x))) + (1.4384*sin(54y)*LegendreP(61, 54, cos(x))) + (−0.7873*sin(36y)*LegendreP(46, 36, cos(x))) + (1.6302*sin(18y)*LegendreP(34,
    18, cos(x))) + (1.4897*sin(16y)*LegendreP(33, 16, cos(x))) + (3.0349*sin(8y)*LegendreP(30, 8, cos(x))) + (−1.3077*sin(2y)*LegendreP(29, 2, cos(x))))
```



Those graphs are unfortunately not what we were expecting. They were supposed to be very messy, but they are actually not.

We will not get their zero-set graphs, since they will not be what we are looking for.

We have yet to figure out what the problem is with Maple and the computations.

7 Conclusion

We have found and graphed many eigenfunctions of $\hat{H} = -\frac{1}{k^2}\frac{d^2}{dx^2} + x^2$, for different values of k. Also, for all these eigenfunctions $V(x)$, we had $V(-1) = V(1) = 0$.
We measured the accuracy of each of these functions and observed that they were all very accurate.

With those functions, using a fact established at the beginning of this document, we can very easily produce and graph eigenfunctions for the Grushin operator $\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}$.

Even though we almost finished the second part of the project, we unfortunately realised that the graphs we obtained with Maple were not what we expected. For this reason, we didn't try to get the zero sets, since they would not be interesting to observe.
However, it would be interesting to figure out what caused these problems and solve them to finally obtain what we wanted.

Despite this unfortunate situation, we have still coded a program that outputed us many couples (l, m) to be graphed as linear combinations of Legendre functions.